Five-Colour Theorem and Beyond

Bojan Mohar

Simon Fraser University

Coast-To-Coast Seminar
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Four-Colour Theorem

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Four-Colour Theorem and its controversy

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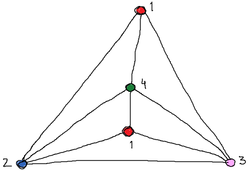


[1] K. Appel, W. Haken (1977), Every Planar Map is Four Colorable Part I. Discharging, Ill. J. Math. 21: 429–490. K. Appel, W. Haken, J. Koch (1977), Every Planar Map is Four Colorable Part II. Reducibility, Ill. J. Math. 21: 491–567.

[2] K. Appel, W. Haken (1989), Every Planar Map is Four-Colorable, AMS.

[3] N. Robertson, D.P. Sanders, P. Seymour, R. Thomas (1997), The Four-Colour Theorem, JCTB 70: 2–44.

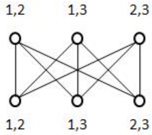
List colouring

List of admissible colours: $\forall v \in V(G): L(v) \subset \{1,2,3,\dots\}$

List colouring ϕ:

- $\forall v \in V(G): \quad \phi(v) \in L(v)$
- $\forall uv \in E(G): \quad \phi(u) \neq \phi(v)$

k-list-colouring: L-colouring where each vertex has k admissible colours.



Not every bipartite graph is always 2-list-colourable (i.e. not 2-choosable).

5-Colour Theorem

5-Colour Theorem (Thomassen, 1994).
Planar graphs are 5-list-colourable.

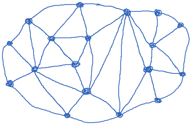
Remark: Not all planar graphs are 4-choosable (Voigt, 1993).

Book proof

Theorem. G near-triangulation

- $|L(a)|, |L(b)| \geq 1$ for two adjacent vertices on the infinite face
- $|L(u)| \geq 3$ for other vertices on the infinite face
- $|L(u)| \geq 5$ for the vertices not on the infinite face.

Then G is L-colourable.



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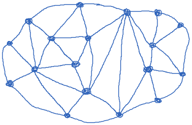
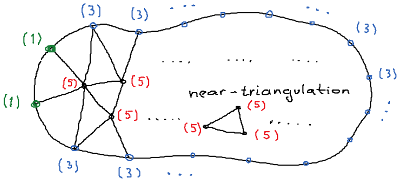
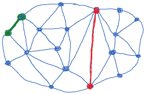


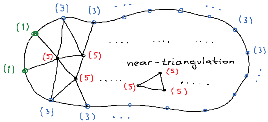
A near-triangulation



Number of admissible colors

Case 1 - Chords





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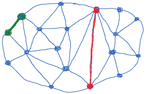

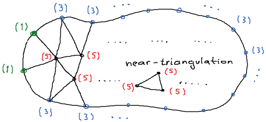

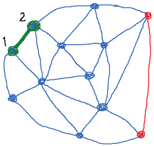

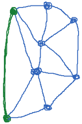

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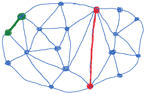
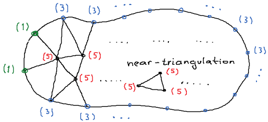
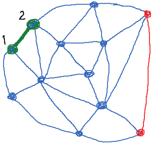
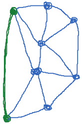
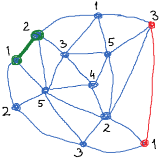
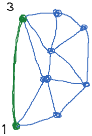
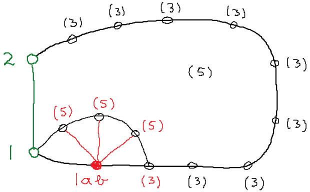







Case 2 - No chords



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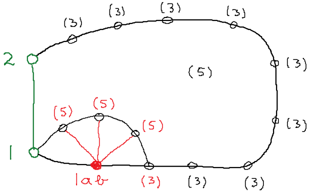
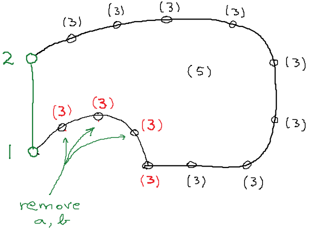
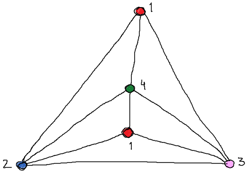




Precolouring extension

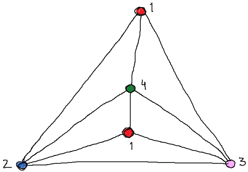


Two red vertices are coloured the same in every 4-colouring.

Question (Thomassen, 1990's)

- Precoloured vertices X of a planar graph, $dist(x, y) \geq 100$, $\forall x, y \in X$. Can we extend to a 5-colouring?

Precolouring extension



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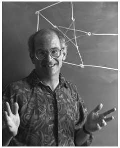
Question (Thomassen, 1990's)

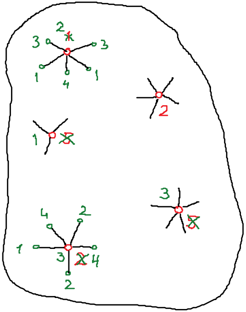
- Precoloured vertices X of a planar graph, $dist(x,y) \geq 100$, $\forall x,y \in X$. Can we extend to a 5-colouring?
- Precoloured vertices X of a planar graph, $dist(x,y) \geq 10^{10}$, $\forall x,y \in X$. Can we extend to a 5-colouring?

Not for 4-colouring!!



Theorem (Albertson, 1998).

G 4-colourable, $X \subset V(G)$ at distance ≥ 4 from each other.
Then every 5-colouring of X extends to a 5-colouring of the whole graph.



Albertson's Conjecture

Curious about the list-coloring version, Mike Albertson asked a question that became known as *Albertson's Conjecture*.

Question (Albertson, 1998).
G planar, $X \subset V(G)$ at distance $\geq 10^{10}$ from each other, $|L(v)| \geq 5$, $\forall v \in V(G)$.
Is it true that every L-colouring of X extends to an L-colouring of the whole graph.

Some other extensions

There are other relaxations where 5-coloring results exist:

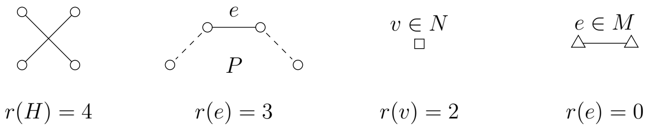
- Graphs drawn with crossings (far apart from each other).
- Precolored vertices (far apart).
- Locally planar graphs (arbitrary surfaces), no short non-contractible cycles.
- Longer precolored path on the infinite face (3 vertices).
- Precolored edges or triangles ...

Recent results for list-colourings

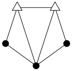
- Locally planar graphs (arbitrary surfaces), no short non-contractible cycles (DeVos, Kawarabayashi, M., 2008).
- Crossings at distance ≥ 19 from each other (Dvorak, Lidicky, M.)
- Albertson's Conjecture (Dvorak, Lidicky, M., Postle).
- Any combination of the above ingredients.

Crossings far apart

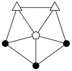
- Precoloured path $P \subset H$ on the outer face with up to 4 vertices
- Special subgraphs: Crossings, vertices with only 4 colours, 3-3 edges, far apart
  $dist(A, B) \geq r(A) + r(B) + 7$
- Every obstruction is colourable
- 1/3/4/5 available colours.



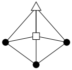
Obstructions



O_{M1}



O_{M2}



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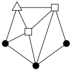
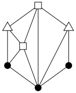
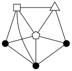


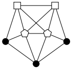
O_{N2}



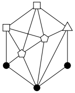
O_{N3}



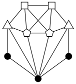
O_{C1}



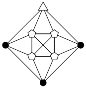
O_{C2}



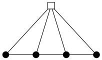
O_{C3}



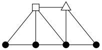
O_{C4}



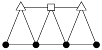
O_{C5}



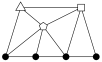
O_{P1}



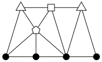
O_{P2}



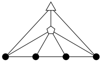
O_{P3}



O_{P4}



O_{P5}



O_{P6}

Some more ingredients

T-critical w.r.t. L:
$T \subset G$, s.t. $\forall e \in E(G) \setminus E(T) : \exists$ L-colouring of T that extends to $G - e$ but does not extend to G.

Lemma. If G is T-critical (w.r.t. L), then

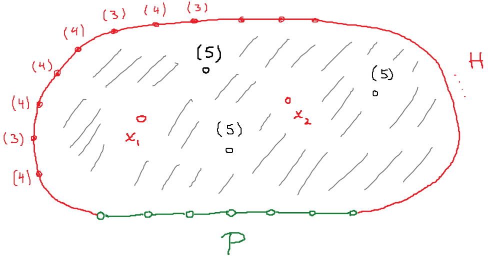
$$\omega_{T,L}(G) \leq |T| - \frac{|V(G) \setminus V(T)|}{2|T|+2} - \frac{9}{2}.$$

where $\omega_{T,L}(G)$ "counts" large faces and vertices in T with ≥ 4 colors.

Corollary. T connected subgraph of G, vertices in $V(G) \setminus V(T)$ have ≥ 5 available colours (but no restrictions on T). If G is not L-colourable, then G contains a subgraph F on $\leq 72|V(T)|^2$ vertices that is not L-colourable.

Basic setup

- G plane graph, outer cycle H
- $P \subset H$ a path in H that is precoloured
- X (other) precoloured vertices, far apart
- 5 available colours for $V(G) \setminus (V(H) \cup X)$
- $3/4/5$ available colours for $V(H) \setminus (V(P) \cup X)$, no 3-3 edges!

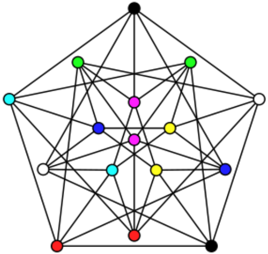


Theorem. (Extending a colouring of a path)
If G is P-critical w.r.t. L, then $\omega_{P,L}(G) \leq |P| - 3$.

This generalises Thomassen's theorem for $|P| = 2$ since $\omega_{P,L}(G) \geq 0$.

Outline of the proof of Albertson's Conjecture

- Reduction to the case with $|X| = 1$: Using reduction techniques from our earlier paper (crossings far apart) and assuming we have a minimal counterexample.
- $P \subset H$ a path in H that is precoloured. Since $\omega_{P,L}(G) \leq |P| - 3$ and H has no 3-3 edges, we have a bound on the length of H (every vertex of H with list of size 4 contributes to ω).
- Reduction to the case when $|X| = 1$ and P has length ≤ 2. Part of the proof is computer supported.
- Here we characterise possible obstructions (infinite families) and then show that we can stay away from them.



Questions?