Optimal control of networks: energy scaling and open challenges

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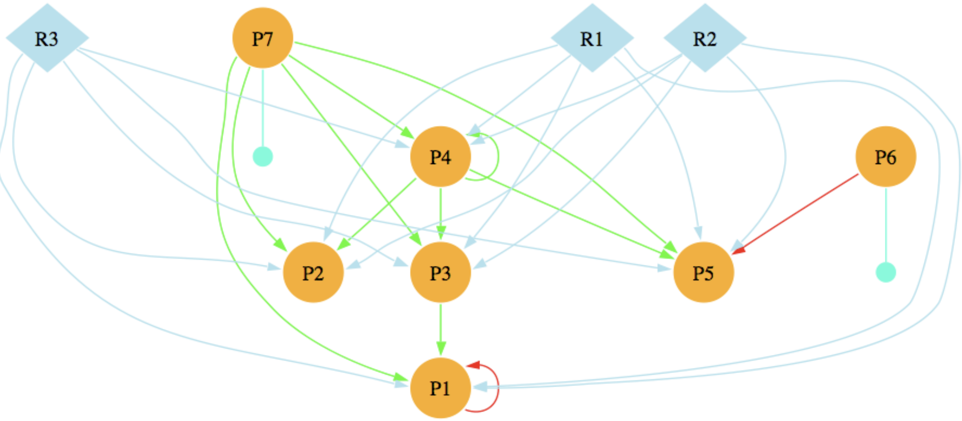
Simon Fraser University, October 13th 2017

Complex Networks

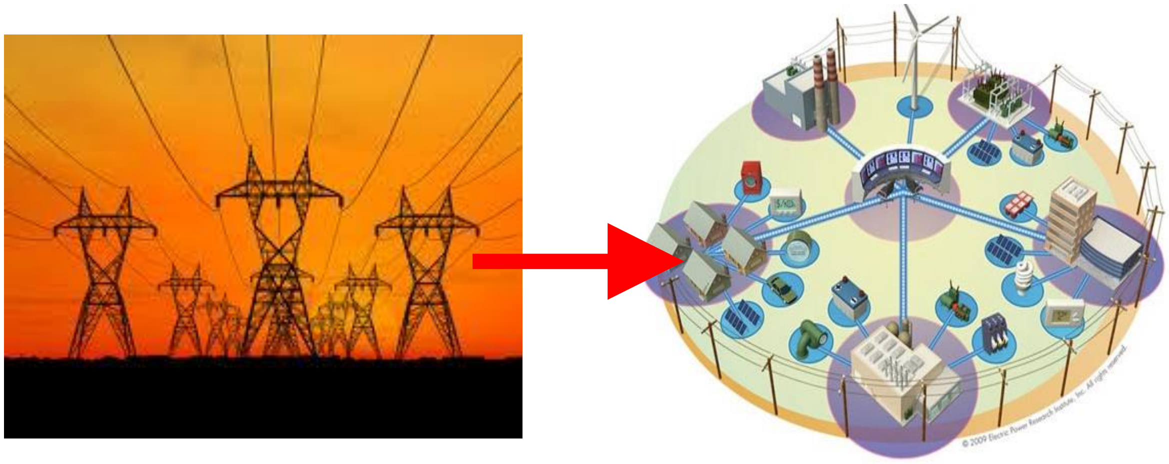
- A large variety of natural and artificial systems can be represented in terms of networks. For instance:
  - Biological networks



A gene-regulatory network

Complex Networks

- A large variety of natural and artificial systems can be represented in terms of networks. For instance:
  - Biological networks
  - Power networks



Complex Networks

- A large variety of natural and artificial systems can be represented in terms of networks. For instance:
  - Biological networks
  - Power networks
  - Computer Networks

Map of the Internet at the AS level (US)

Complex Networks

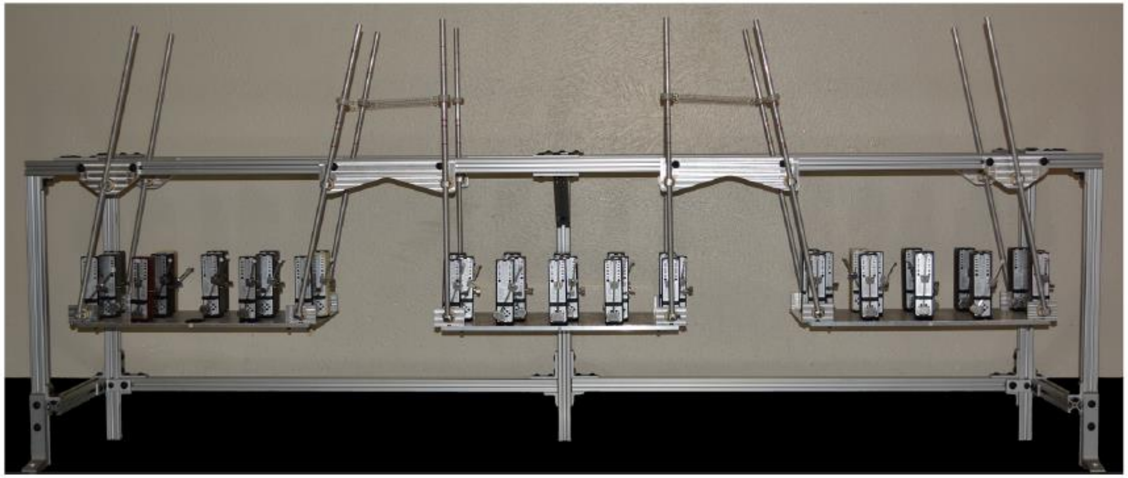
- A large variety of natural and artificial systems can be represented in terms of networks. For instance:
  - Biological networks
  - Power networks
  - Computer Networks
  - Traffic Networks

Complex Networks

- A large variety of natural and artificial systems can be represented in terms of networks. For instance:

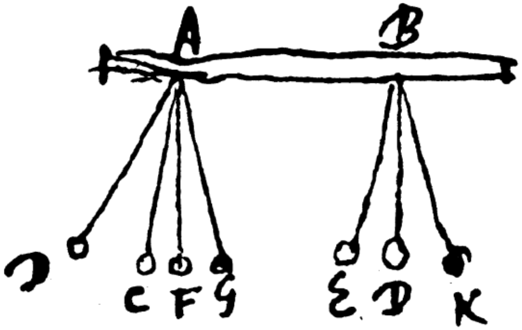
  - Biological networks
  - Power networks
  - Computer Networks
  - Traffic Networks
  - Mechanical Networks

ISSUES:
- **MODELING**
- **DYNAMICS**
- **CONTROL**

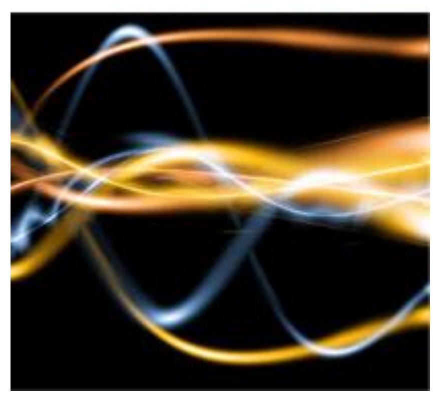


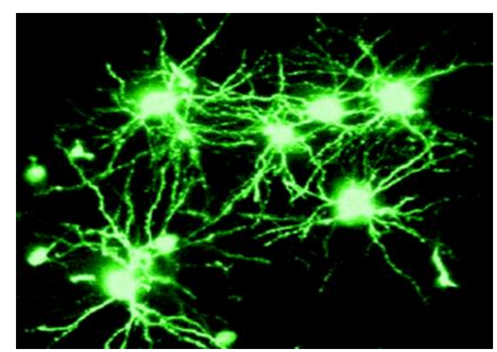
Dynamics of Complex Networks: Synchronization

- Huygens (1665): synchronization of two weakly coupled clocks

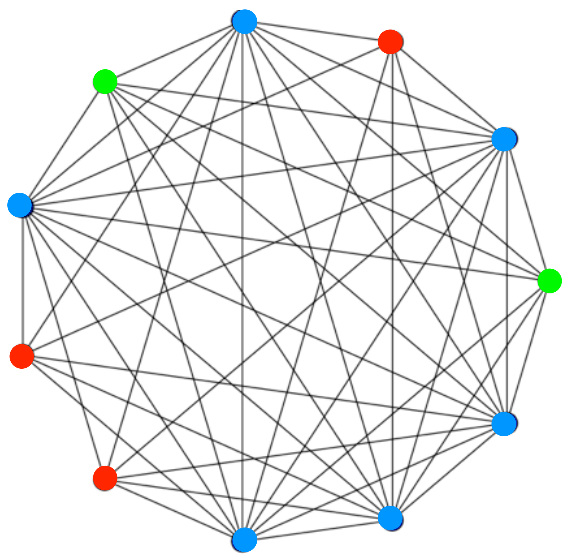


- Current applications:
  - Epidemics
  - Flocking
  - Secure communications
  - GPS
  - Clock synchronization
  - Neural networks





Cluster Synchronization



$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{x}_j)$$

- Identify the clusters?
- Are the clusters stable?
- Complex (large) networks
- Any dynamics (fixed pt, periodic, quasiperiodic, chaotic

L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, T. E. Murphy, R. Roy, "Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries", *Nature Communications*, 5, 4079 (2014).

F. Sorrentino, L. M. Pecora, A. M. Hagerstrom, T. E. Murphy, R. Roy, "Complete characterization of stability of cluster synchronization in complex dynamical networks", *Science Advances* 2, e1501737 (2016).

Control of Networks (1)

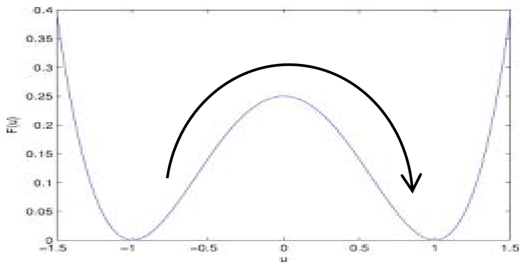
- Power Grid Dynamics: maintaining frequency of generators in the presence of perturbations

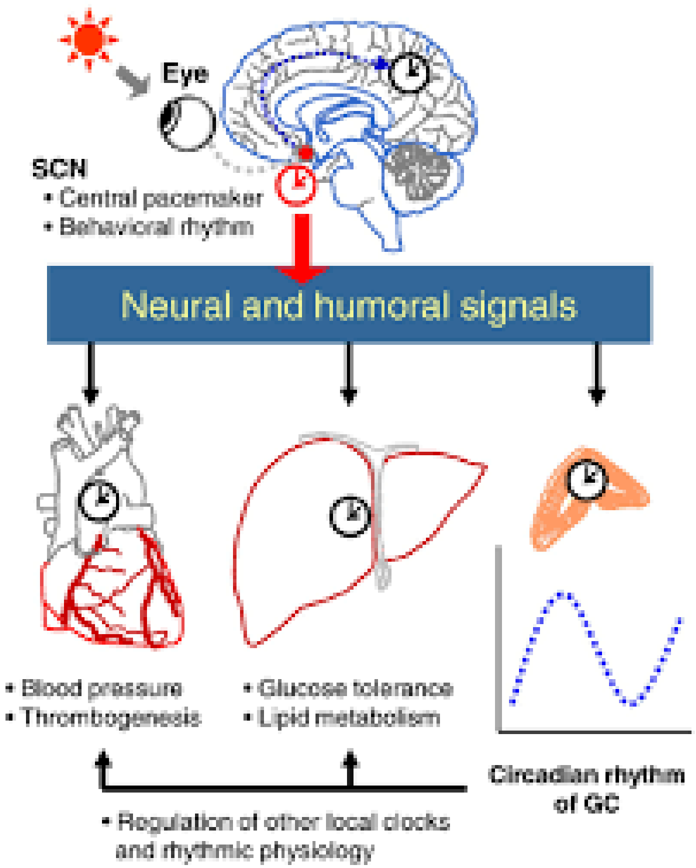
Control of Networks (2)

- Control of Mammalian Circadian
- Rhythm

- The dynamics is multistable (both fixed points and limit cycles)

- Problem: moving from one attractor to the basin of attraction of another attractor



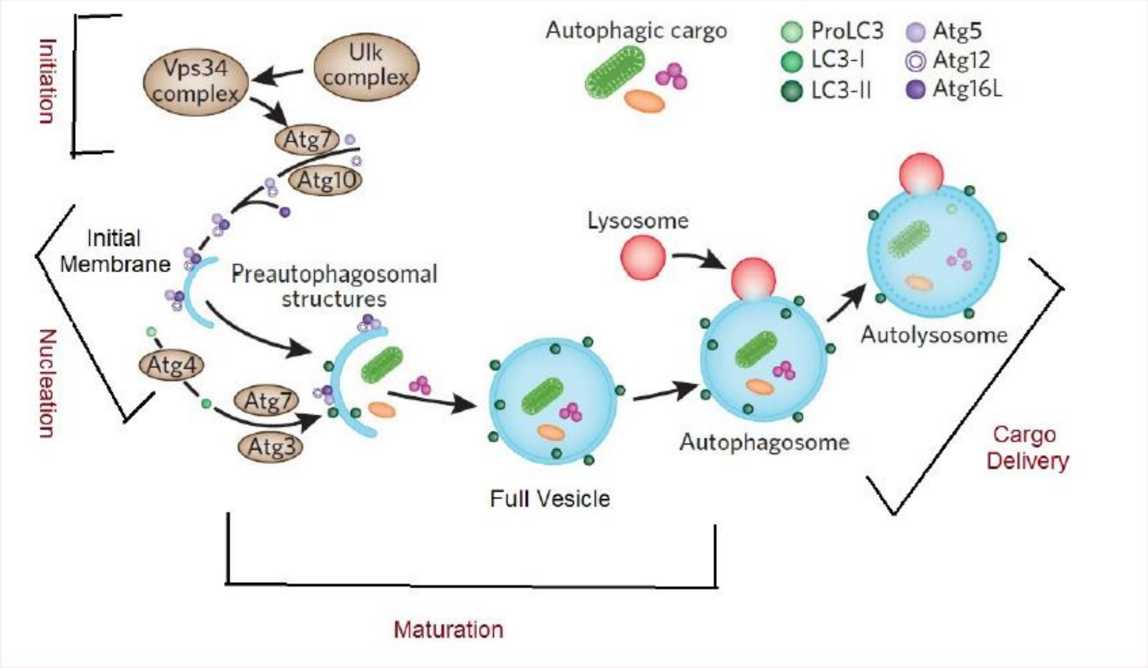


Chung, Son, Kim, Circadian rhythm of adrenal glucocorticoid: Its regulation and clinical implications

Control of Networks (3)

- Control of Autophagy in a single cell

◆ Consider the Autophagic system in a single cell,

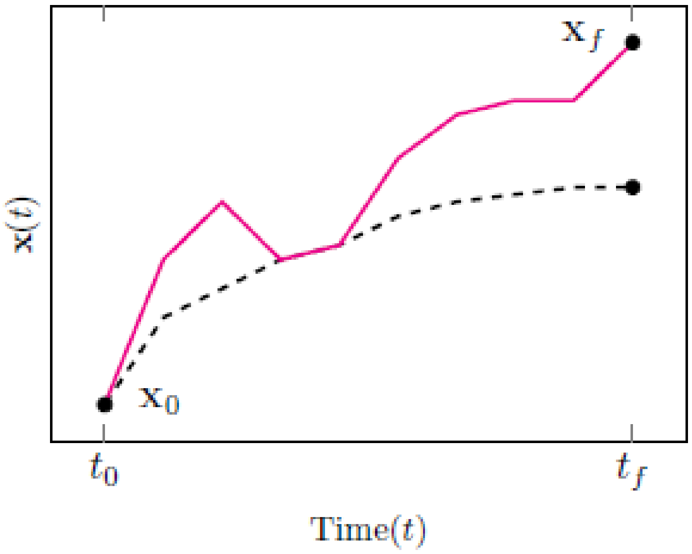


One option: Optimal Control

Controllability

Consider the continuous time system,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + B\mathbf{u}(t)$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{x}(t_f) = \mathbf{x}_f$$



One option: Optimal Control

Control Energy

Control Energy,

$$J = \int_{t_0}^{t_f} \mathbf{u}(t)^T \mathbf{u}(t) dt$$

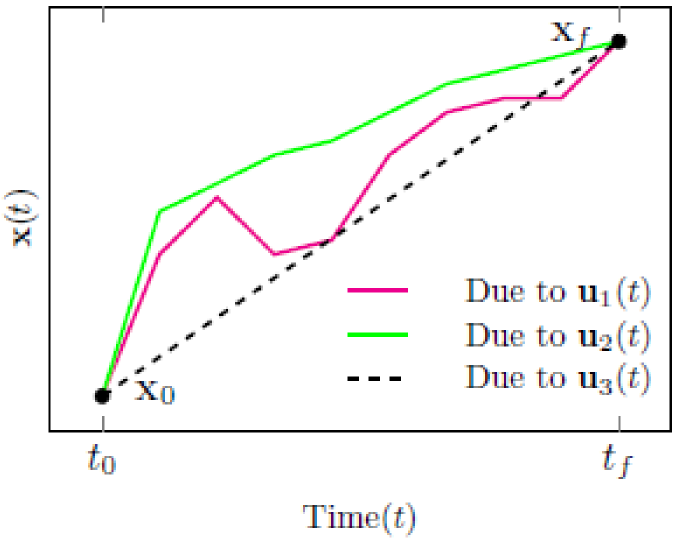
Optimal Control Input

Optimal Control Input = $\mathbf{u}^*(t)$.

Optimal Control Energy

Optimal Control Energy,

$$J^* = \int_{t_0}^{t_f} \mathbf{u}^*(t)^T \mathbf{u}^*(t) dt$$



UNM

Issues

•The dynamics of complex networks is nonlinear

•Control of nonlinear systems is difficult!

•Optimal control strategies for nonlinear systems are typically obtained numerically

•Numerical optimal control solutions for large high-dimensional nonlinear systems are computationally expensive

Target Control of Networks

Isaac Klickstein, Afroza Shirin, Francesco Sorrentino

A network is described by two sets:

1. A set of nodes, $\mathcal{V}$ (often these coincide with the states), and
2. A set of edges, $\mathcal{E}$ (these are the linearized dynamical relations between nodes)

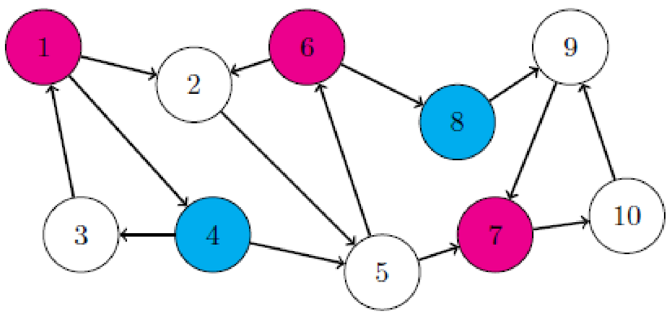


Figure: A 10 Node Network

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k$$

There are three types of nodes:

1. Driver Nodes: These can be directly influenced by our control inputs, u_k, $k = 1, \dots, m$.
2. Target Nodes: These are nodes with a desired final condition.
3. Neither: These are nodes that are neither driven nor targeted.

Target Control of Networks

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A state, $x_i(t)$, $i = 1, \ldots, n$ corresponds to a node $v_i \in \mathcal{V}$.
We define our state vector as,

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} \tag{4}$$

The **adjacency matrix**, $A = \{a_{ij}\}$, contains the **edges** $\in \mathcal{E}$ where if $a_{ij} \neq 0$, the state of v_j affects v_i.

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Target Control of Networks

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To start, we consider all nodes as target nodes.
We define the **control energy** as,

$$E = \int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt \tag{5}$$

The optimization problem is:

$$\begin{aligned} \min_{\mathbf{u}(t)} \quad & J = \frac{1}{2}E = \frac{1}{2}\int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt \\ \text{such that} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ & \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f \end{aligned} \tag{6}$$

$J(\mathbf{x}(t), \mathbf{u}(t))$ is the **cost function**, or penalty function.

Target Control of Networks

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The solution is,

$$\mathbf{u}^*(t) = B^T e^{A^T(t_f - t)} W^{-1} \beta \tag{7}$$

where,

$$W = \int_{t_0}^{t_f} e^{A(t_f - \tau)} B B^T e^{A^T(t_f - \tau)} d\tau, \qquad \beta = \left(\mathbf{x}_f - e^{A(t_f - t_0)} \mathbf{x}_0 \right)$$

W is the controllability Gramian.
More importantly, the minimum energy is,

$$\begin{aligned} E_{\min} &= \int_{t_0}^{t_f} ||\mathbf{u}^*(t)||^2 dt \\ &= \beta^T W^{-1} \beta \end{aligned} \tag{8}$$

Target Control of Networks

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The controllability Gramian tends to be poorly conditioned when,

1. The time interval, $t_f - t_0$ is 'small', or
2. The percentage of nodes which are drivers is small.

Why does the condition of W matter?
Min-Max Theorem

$$E_{\text{min}}^{(\text{min})} \leq \frac{1}{||\beta||^2} \beta^T W^{-1} \beta \leq E_{\text{min}}^{(\text{max})} \tag{9}$$

So,

$$E_{\text{min}}^{(\text{max})} = \frac{1}{\lambda_{\text{min}}(W)} \tag{10}$$

which can be prohibitively large.

Target Control of Networks

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We define an output,

$$\mathbf{y}(t) = C\mathbf{x}(t), \qquad \mathbf{y}(t) \in \mathbb{R}^{p\times 1}, \qquad p \leq n \tag{11}$$

which is a linear combination of the states.

The output can be used to **target** nodes by choosing C such that each row has only one nonzero element.

Problem Statement for MEOCS:

$$\begin{aligned} \min_{\mathbf{u}(t)} \quad & J = \frac{1}{2}E = \frac{1}{2}\int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt \\ \text{such that} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ & \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{y}(t_f) = \mathbf{y}_f \end{aligned} \tag{12}$$

Target Control of Networks

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The optimal control input,

$$\mathbf{u}^*(t) = Be^{A^T(t_f - t)} C^T \left(CWC^T \right)^{-1} \beta \tag{13}$$

The minimum energy is,

$$E_{\text{min}} = \beta^T \left(CWC^T \right)^{-1} \beta = \beta^T W_p^{-1} \beta \tag{14}$$

where W_p is a **minor** of W.

This method reduces the **control space** of the system.

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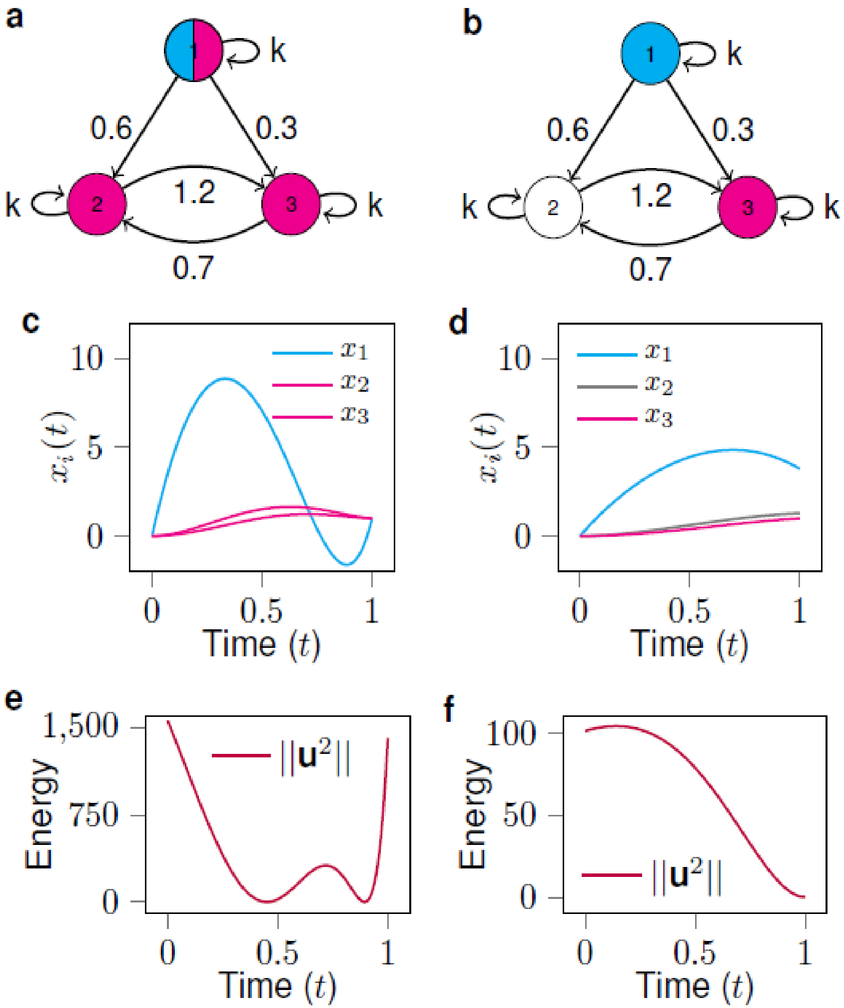

The system on the left uses the MECS formulation to place each node at a final condition. The integral of the energy magnitude curve is $E = 382$.

The system on the right assumes only node three needs to have a final condition, a MEOCS, and is the only node targeted. This time $E = 66.3$, only a sixth of the MECS formulation.

Target Control of Networks

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A four dimensional example:

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$W_p = CWC^T = \begin{bmatrix} W_{22} & W_{24} \\ W_{42} & W_{44} \end{bmatrix}$$

Cauchy Interlacing Theorem:
Proves that the minimum eigenvalue of the minor of a matrix is larger than the minimum eigenvalue of the original matrix.

Target Control of Networks

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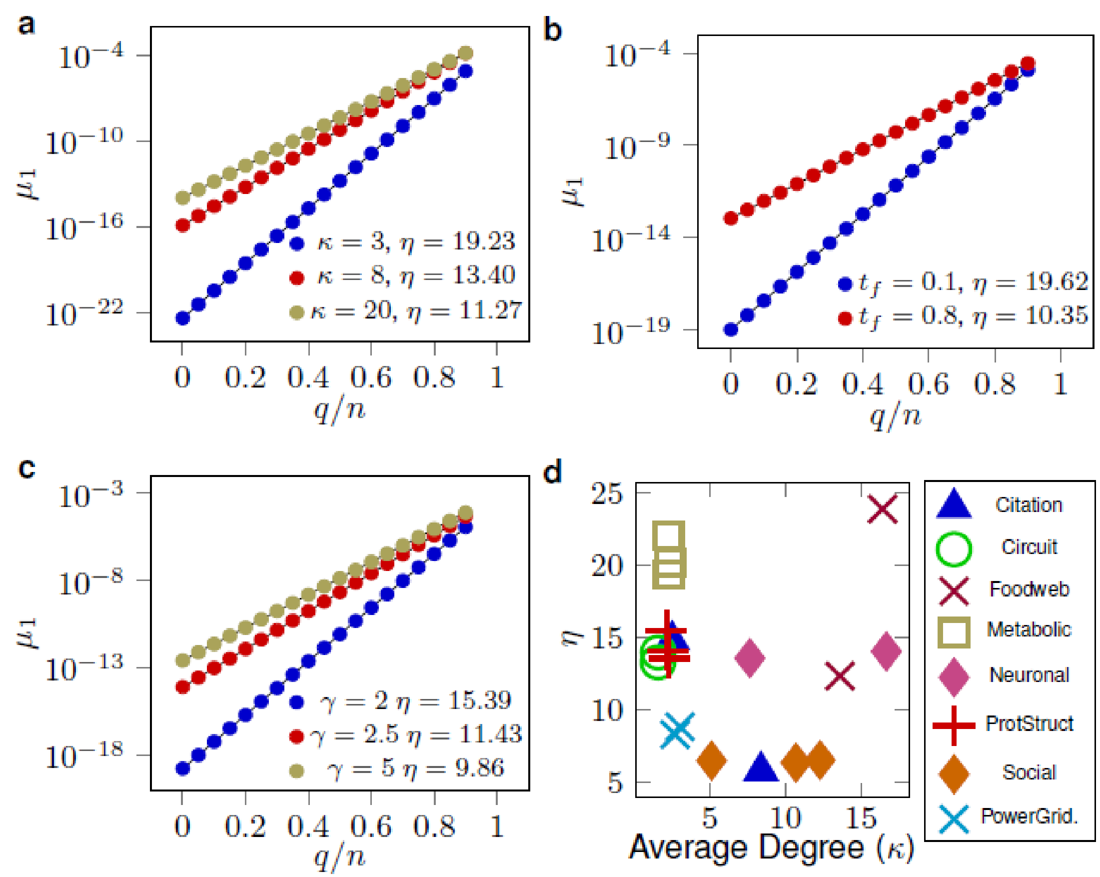

Note that,

$$E_{\text{min}}^{(\text{max})} = \frac{1}{\mu_1}$$

Four Cases:

a) Degree

b) Time Interval

c) Homogeneity

d) Real Datasets

Target Control of Networks

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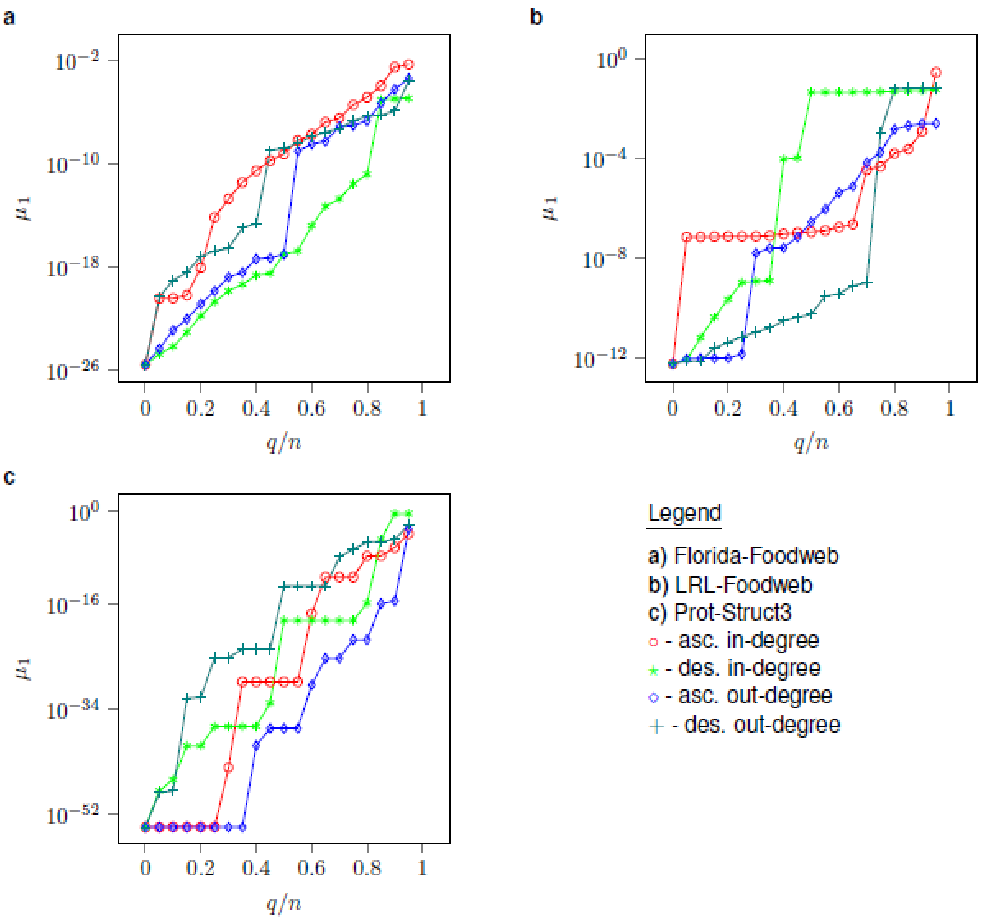

When nodes are chosen by degree, we see much less smooth behavior.

Target Control of Networks

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We define q as the number of non-targeted nodes (the dimension of the complement of the target set).

We define $\mu_1^{(q)}$ as the minimum eigenvalue of $C\tilde{W}C^T$ when $n-q$ nodes are targeted.

$$\mu_1^{(q)} = \mu_1^{(0)} \left(\prod_{i=1}^{q} \eta_i \right) = \mu_1^{(0)} (\eta_{1q})^q \qquad (18)$$

$\eta_{1q} = \left(\prod_{i=1}^{q} \eta_i\right)^{1/q} > 1$ which explains the exponential improvements as the target set is reduced

Target Control of Networks

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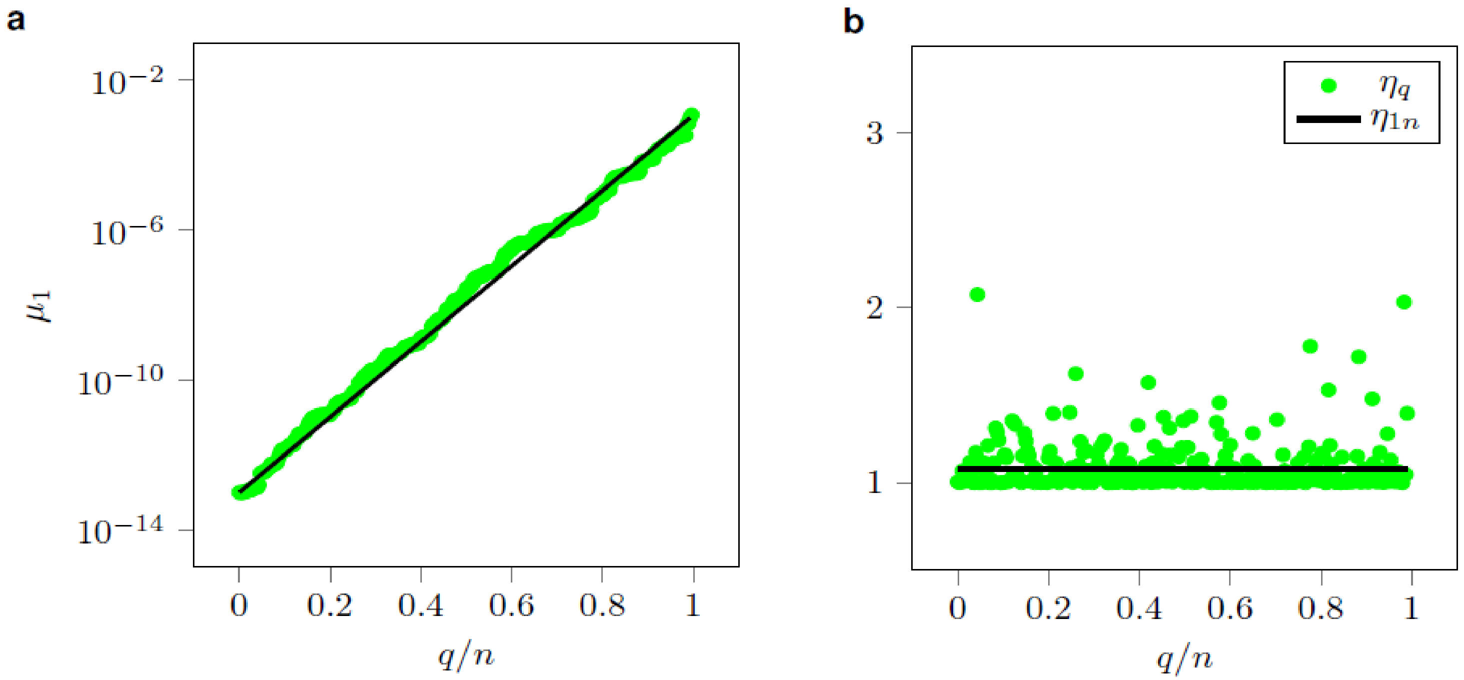

Figure: **a**) The exponential increase of μ_1 as q increases. **b**) The value of η_{1q} is larger than one.

Distance vs # of Targets

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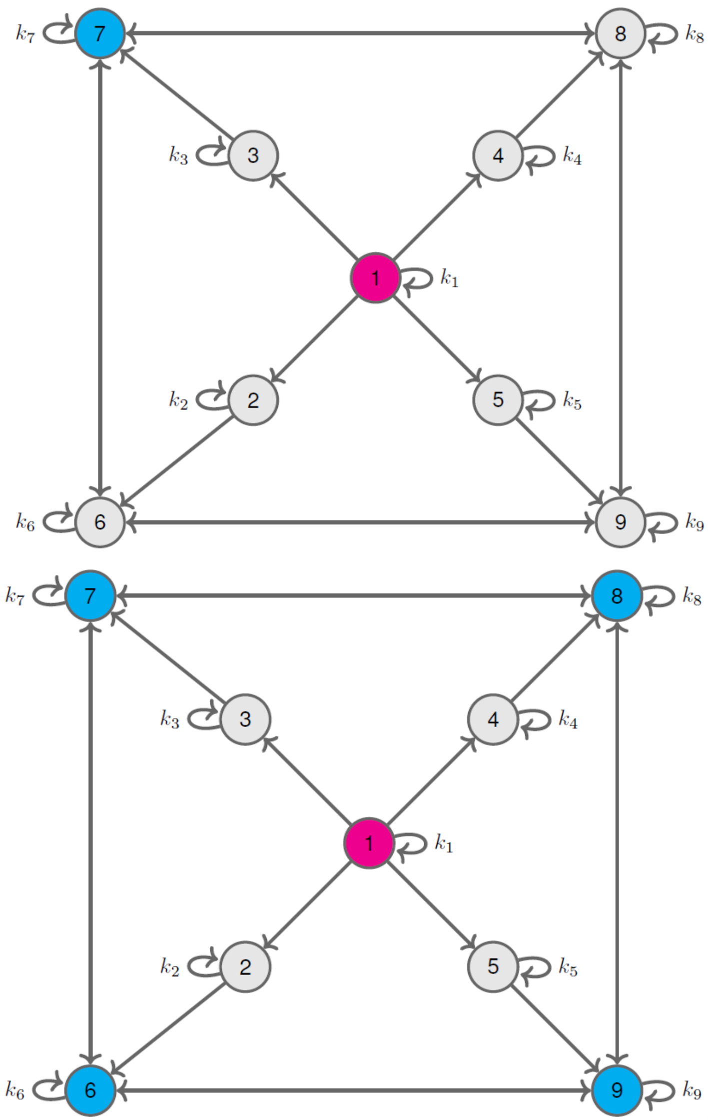

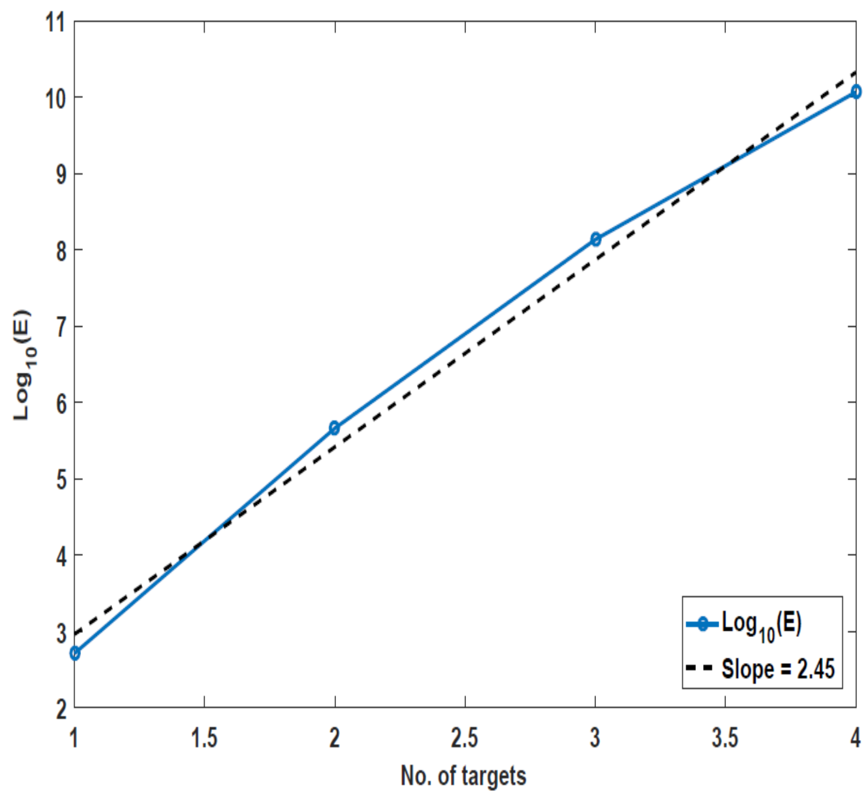

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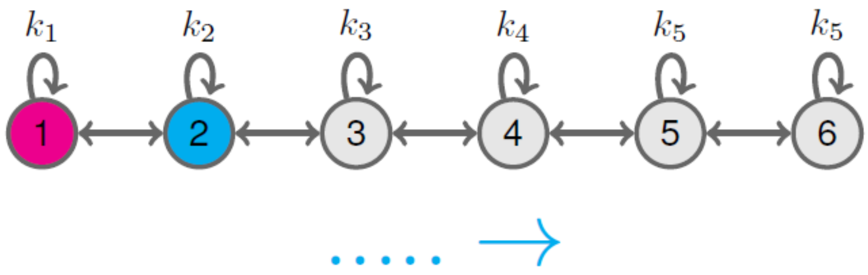

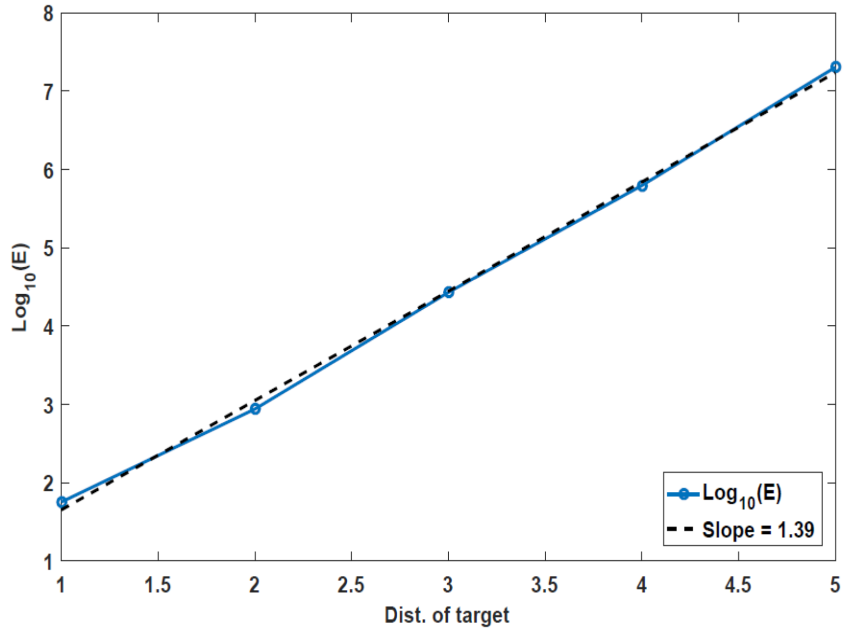

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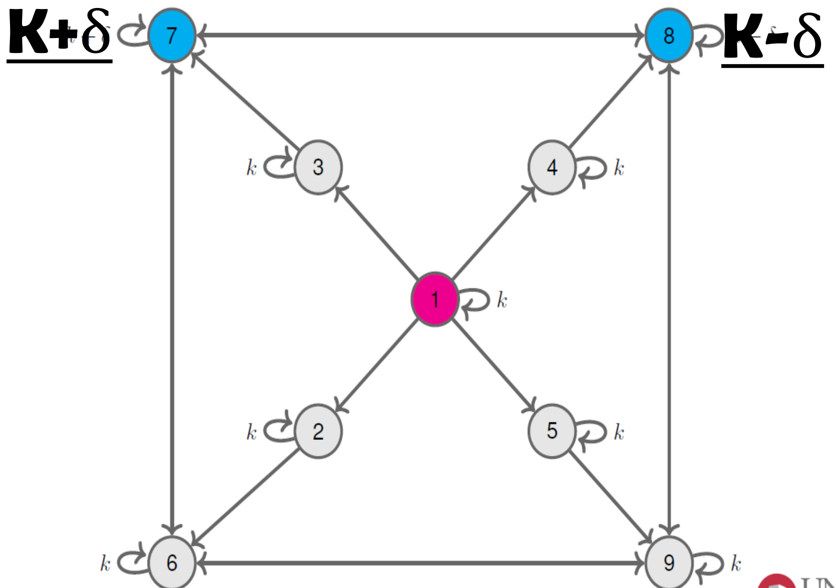
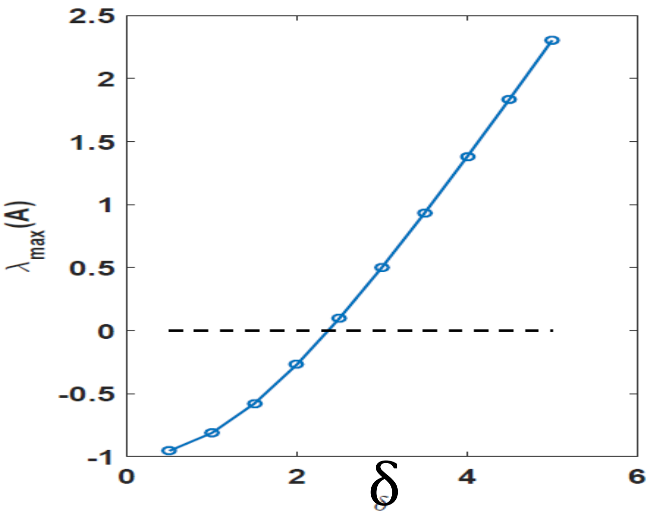
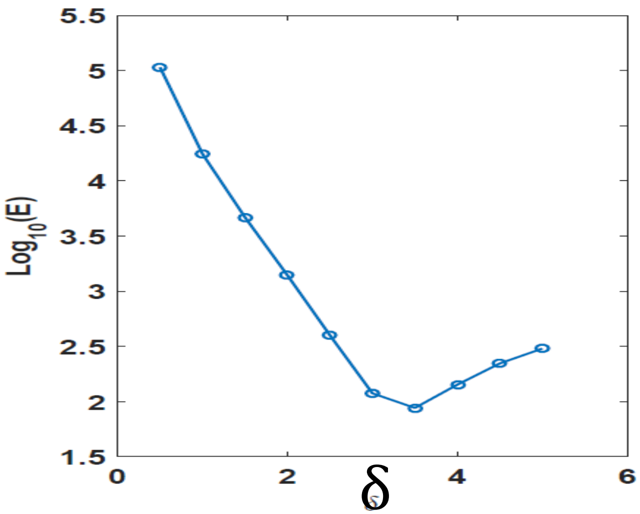
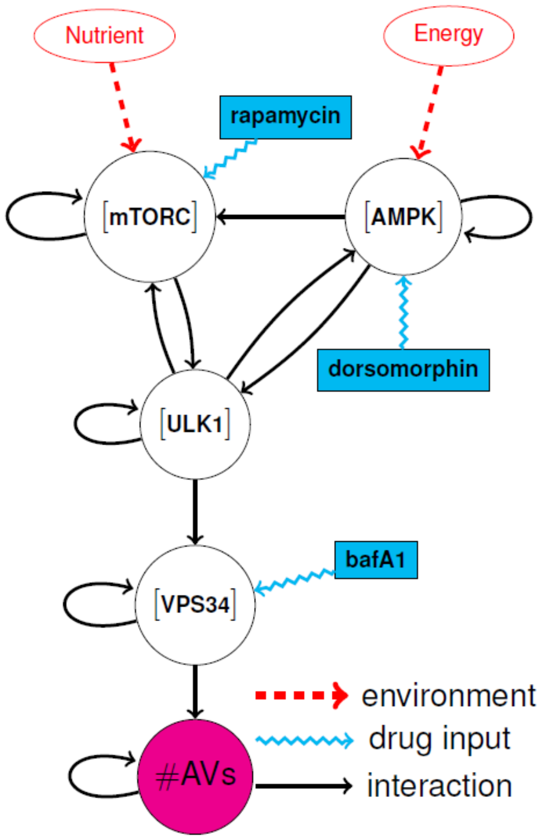






Example: Regulating Autophagy

Isaac Klickstein, Afroza Shirin,Francesco Sorrentino



$$\dot{x}_1(t) = (1-x_1)g_1(1-u_1)(1-u_2) - x_1 h(x_2)h(x_3)$$
$$\dot{x}_2(t) = (1-x_2)h(x_3)(1-u_3) - x_2 h(x_1)$$
$$\dot{x}_3(t) = (1-x_3)k_1(1-u_4) - g_2 x_3 x_2(1-u_5)$$
$$\dot{x}_4(t) = (1-x_4)k_2 h(x_2) - k_3 x_4$$
$$\dot{x}_5(t) = k_4 x_4 - k_5 x_5(1-u_6)$$

x_1: [mTORC]
x_2: [ULK1]
x_3: [AMPK]
x_4: [Vps34]
x_5: # Autophagic Vesicle/10

Regulating autophagy (P. Szymanska, et al. PloS one, 10(3) e0116550, 2015)

Example: Regulating Autophagy

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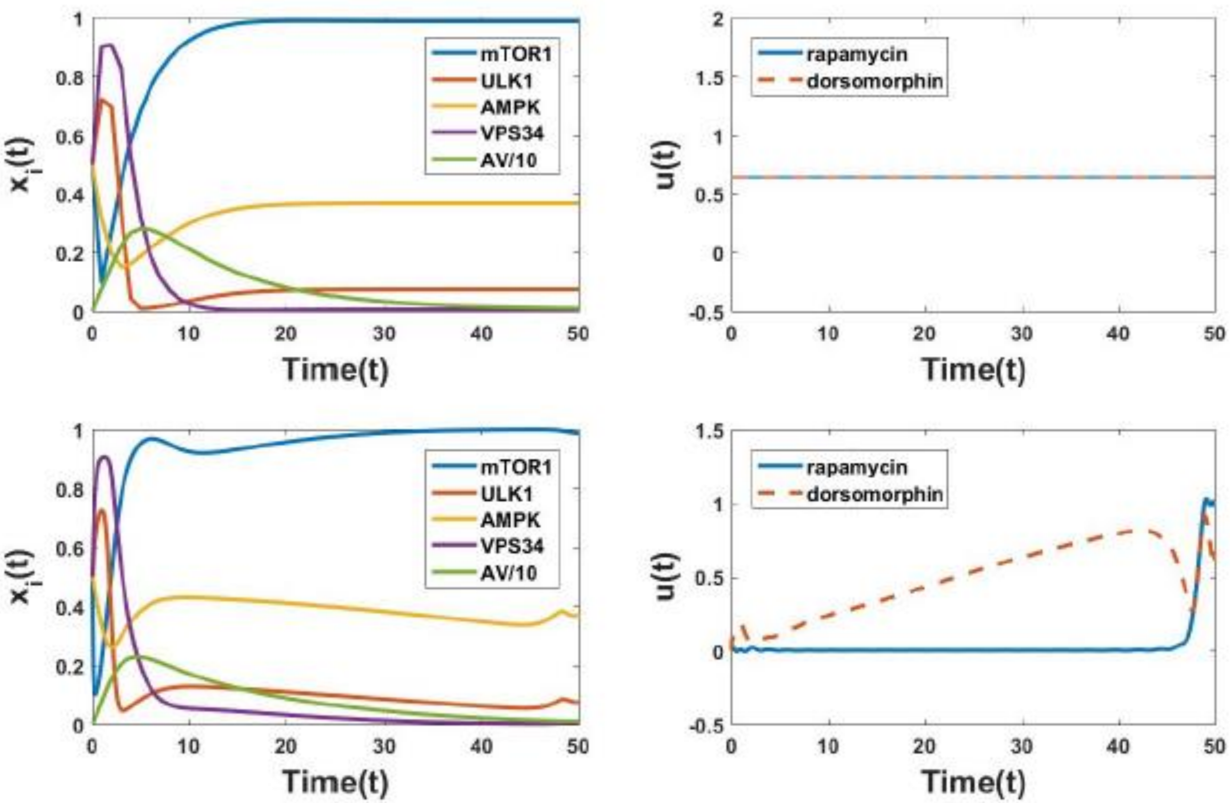


Conclusions

We consider control of large dimensional dynamical networks with applications to biological, technological, ecological systems and so on

By choosing targets, the control energy can be reduced exponentially with respect to the size of the target set.

Optimal control of a nonlinear network (to some nonlocal point) can be achieved by performing a sequence of local optimal controls

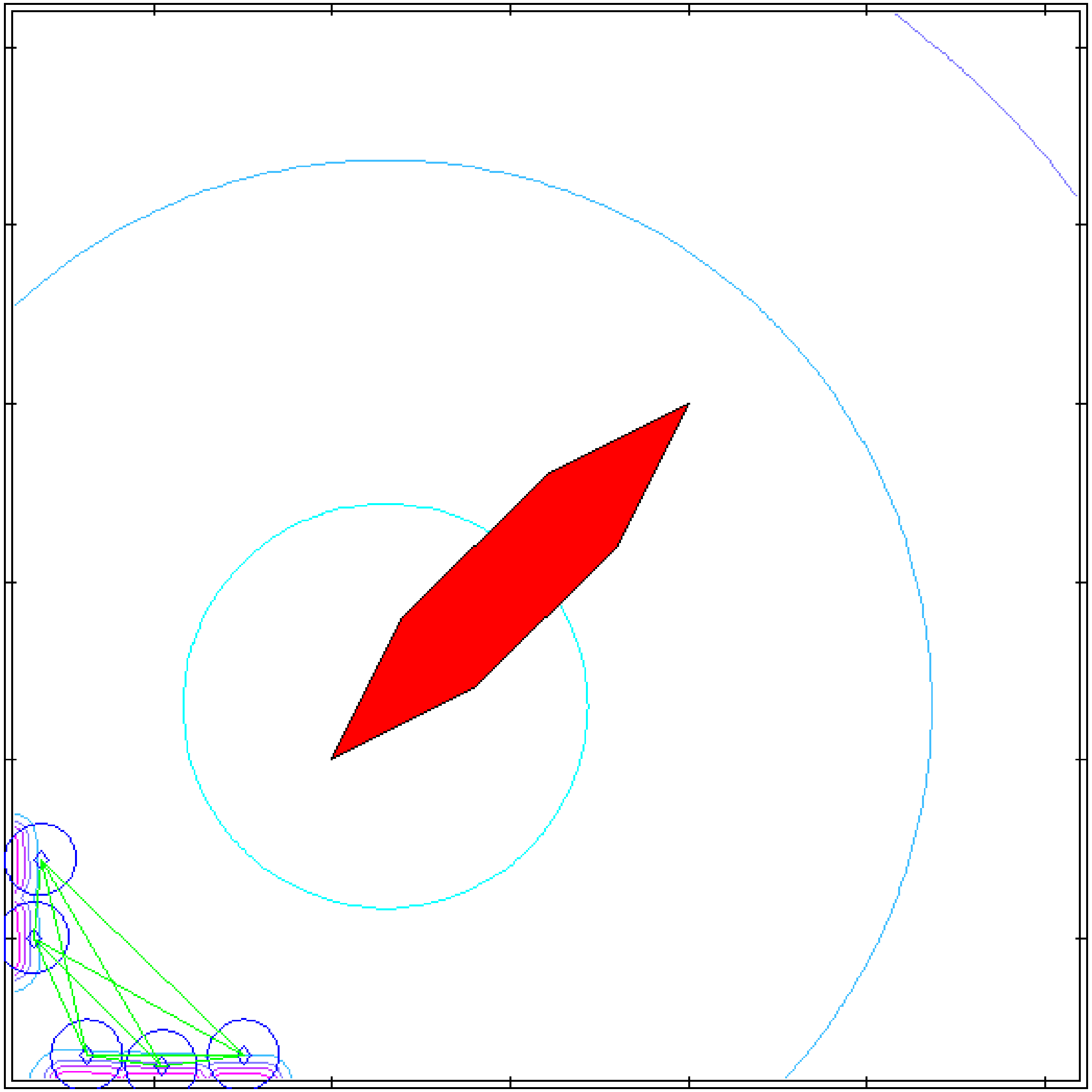
Main References

F. Lo Iudice, F. Garofalo, **F. Sorrentino**, Structural Permeability of Complex Networks to Control Signals, Nature Communications, 6, 8349 (2015).

I. Klickstein, A. Shirin, **F. Sorrentino**, Energy Scaling of Targeted Optimal Control of Complex Networks, Nature Communications, 8, 15145 (2017).

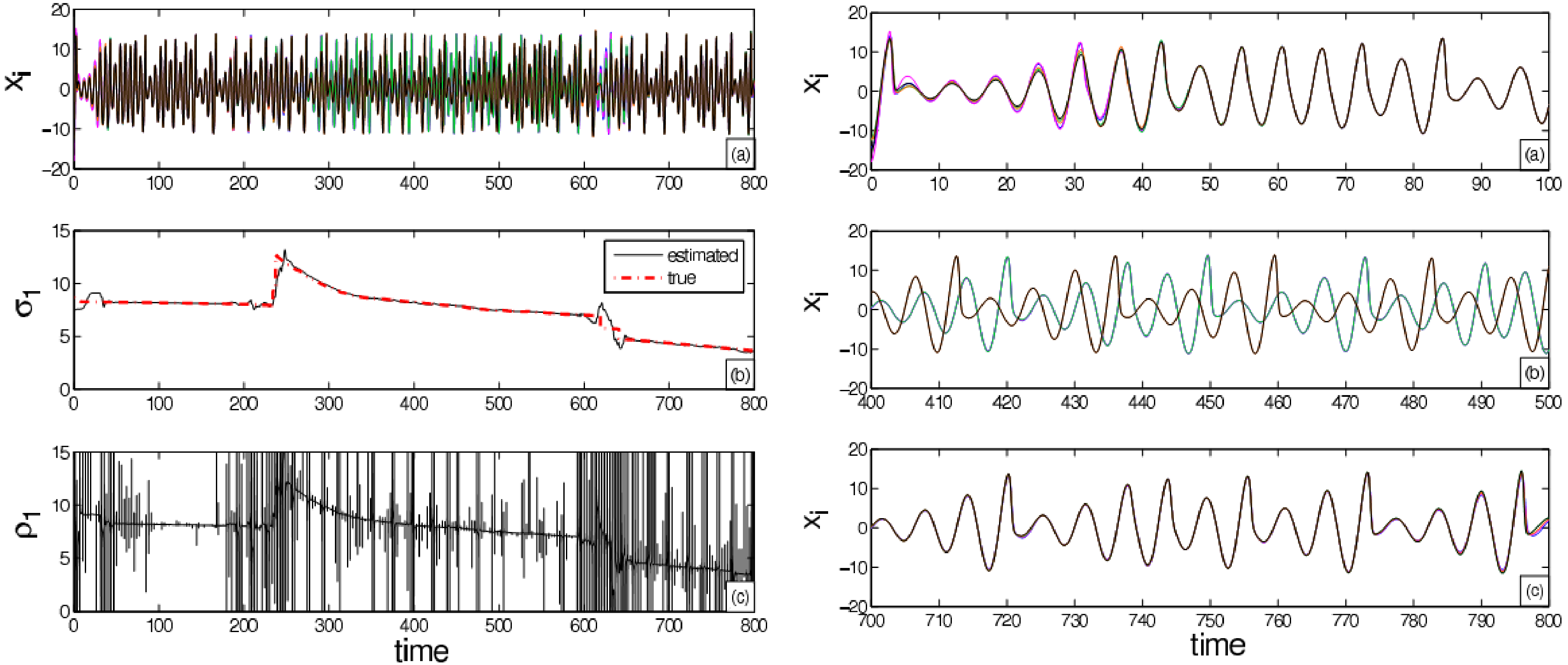
THANKS TO:

Mobile Robots moving in an obstacle-populated environment



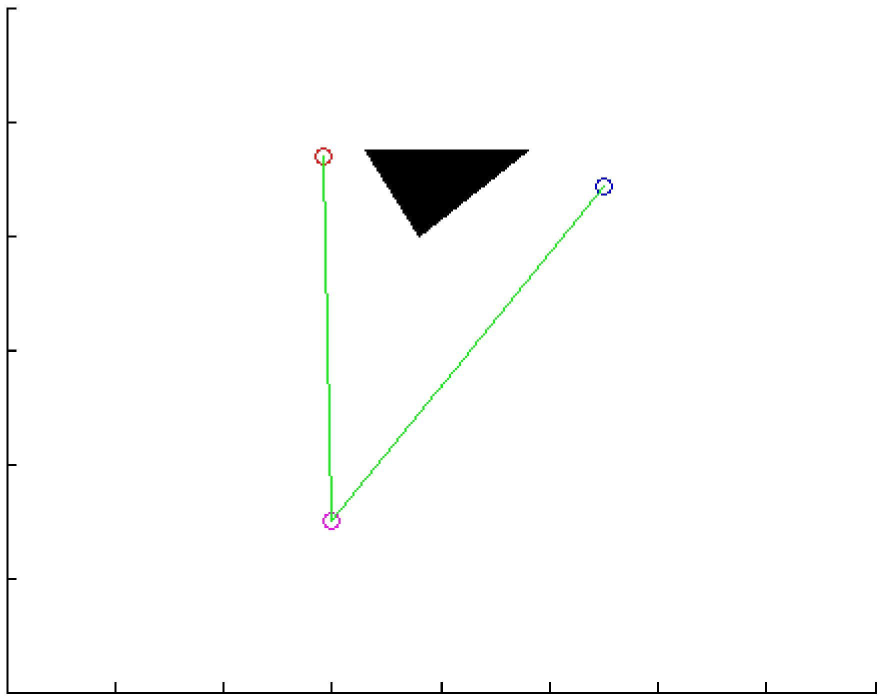
Work in collaboration with N. Bezzo and R. Fierro at UNM

Mobile Robots moving in an obstacle-populated environment



$$\rho_i(t) = \frac{x_i(t)}{r_i(t)}$$

Mobile Robots moving in an obstacle-populated environment



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