Contributors and Co-Authors

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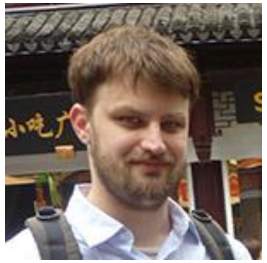
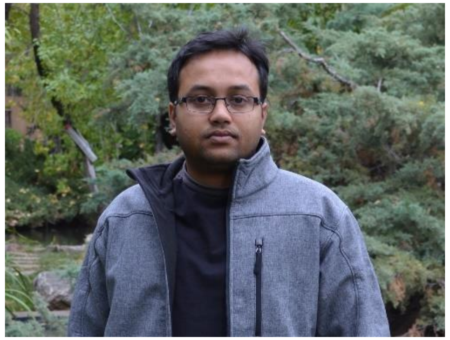
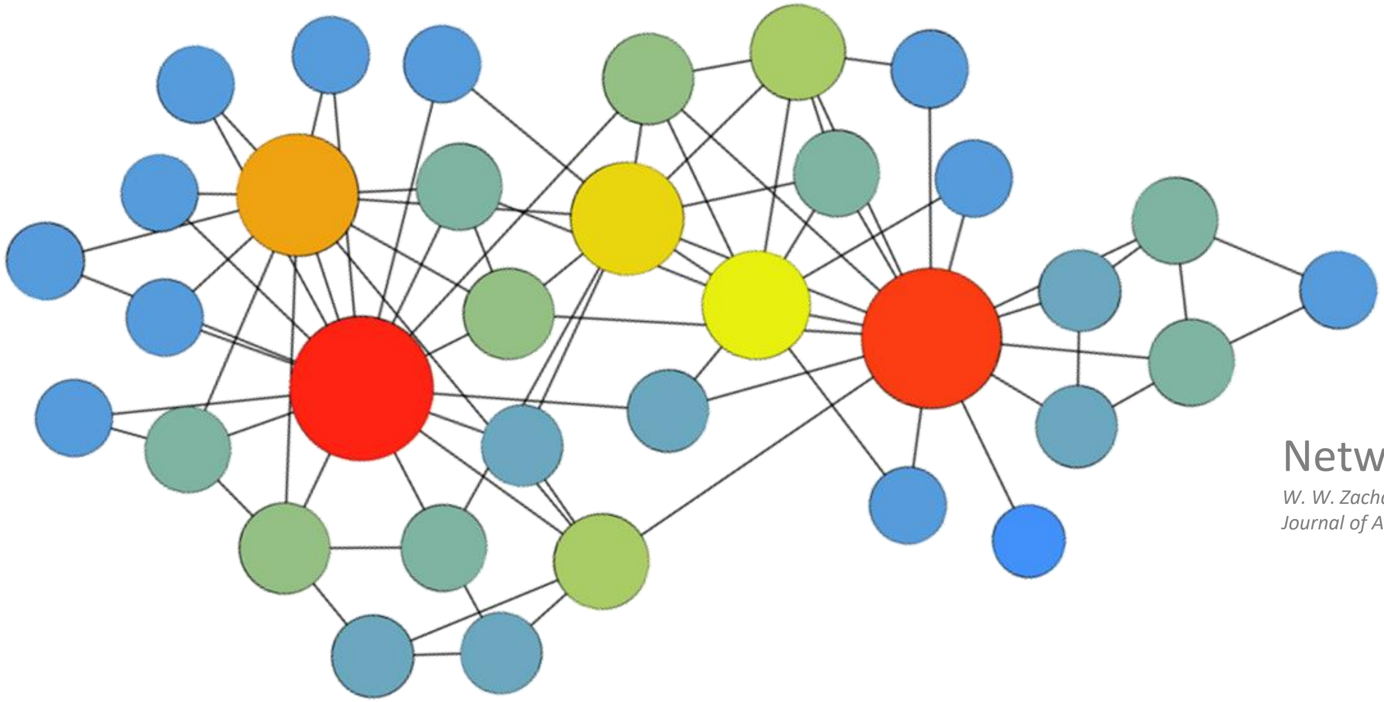
UNM





Complex Networks

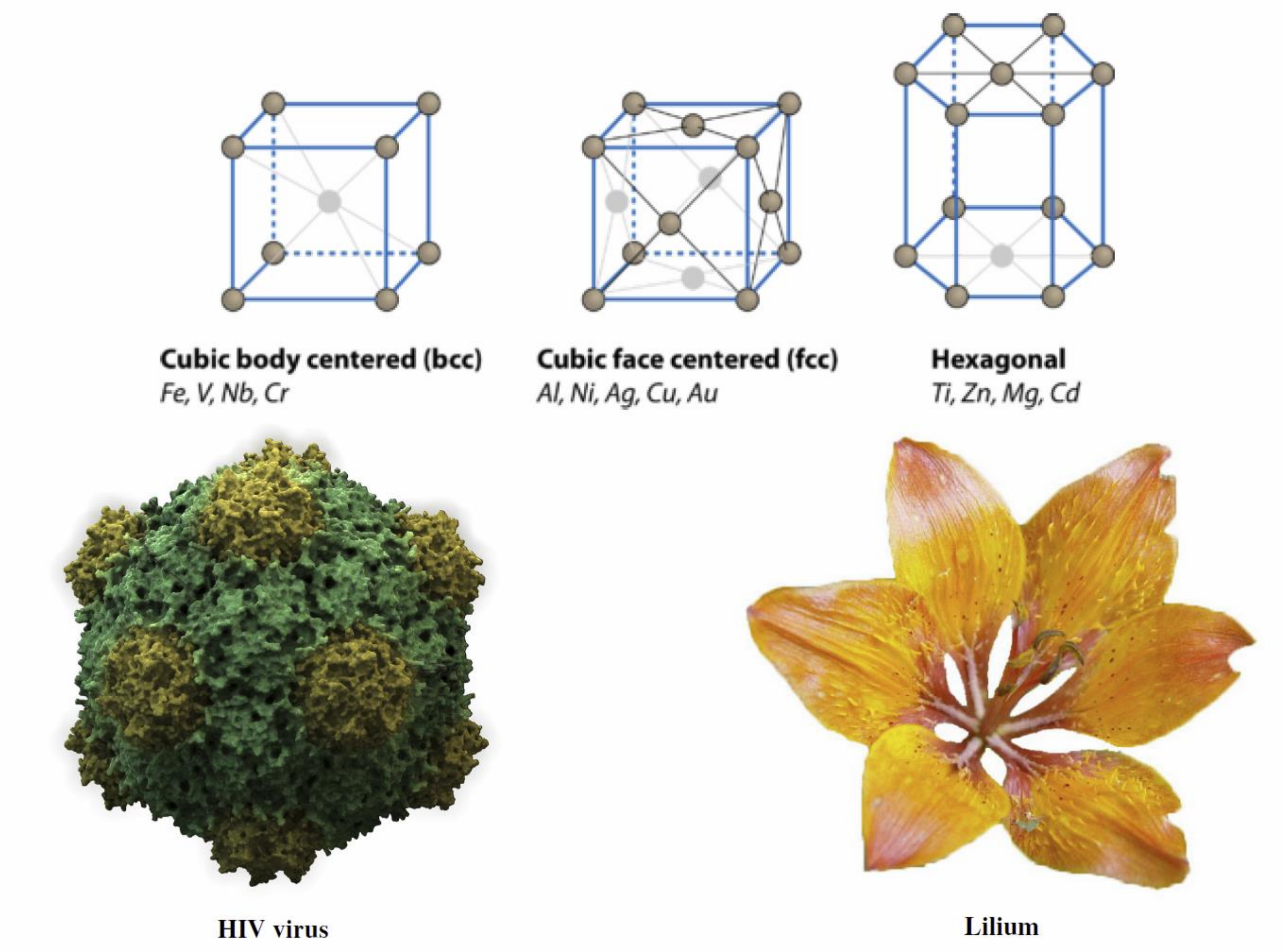
- Network is, in simplest form, a collection of points (vertices or nodes) joined together in pairs by lines (links, edges or bonds). i.e. lattices, random graphs, small world networks, scale-free networks. A network is mathematically expressed by an adjacency matrix.
- A complex network is a graph (network) with non-trivial topological features.



Network: Zachary's karate club

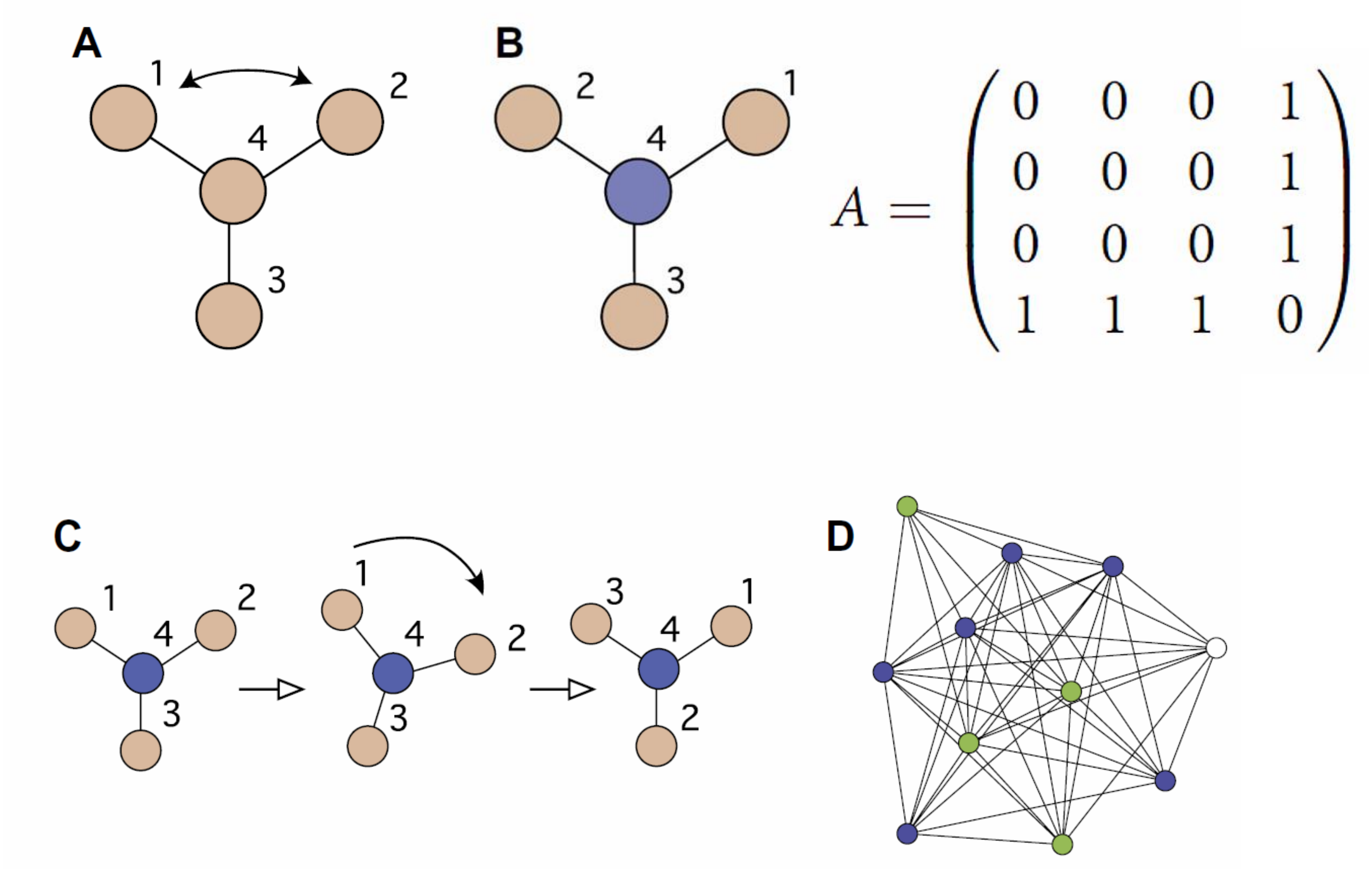
W. W. Zachary, An information flow model for conflict and fission in small groups, Journal of Anthropological Research 33, 452-473 (1977).

Symmetries



Cubic body centered (bcc)
Fe, V, Nb, Cr
Cubic face centered (fcc)
Al, Ni, Ag, Cu, Au
Hexagonal
Ti, Zn, Mg, Cd
HIV virus
Lilium

Symmetries and Clusters



$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Synchronization

The dynamics of the network,

$$\dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{N} A_{ij} H(x_j)$$

An alternate description in case of Laplacian coupling,

$$\dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{N} L_{ij} H(x_j)$$

Chaotic Systems

Sensitivity to Initial Conditions

$$\frac{dx_1}{dt} = \sigma(y_1 - x_1)$$

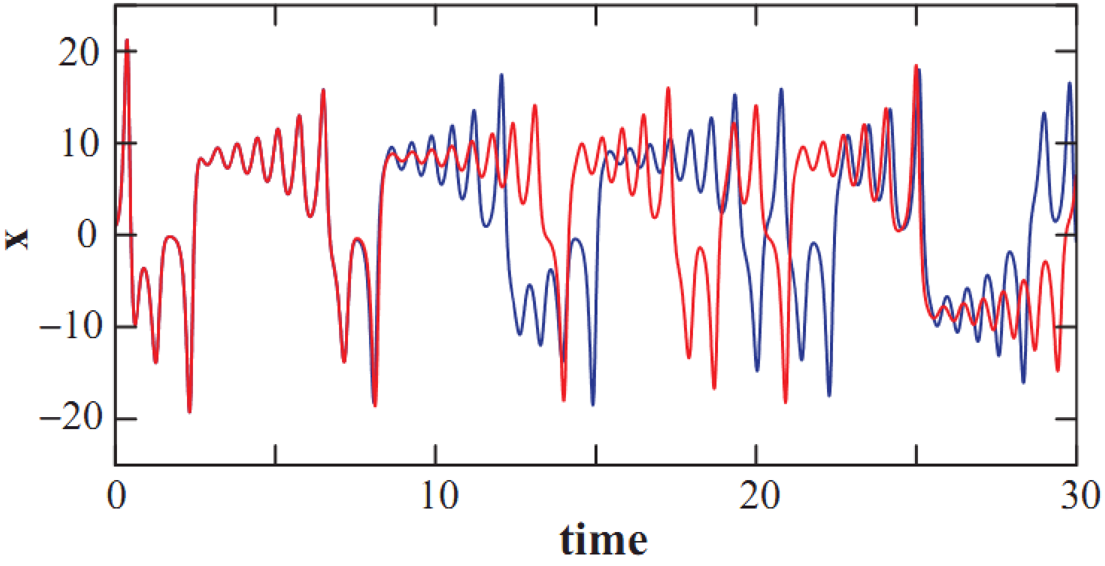
$$\frac{dy_1}{dt} = x_1(\rho - z_1) - y_1$$

$$\frac{dz_1}{dt} = x_1 y_1 - \beta z_1$$

$\sigma = 10, \rho = 28, \beta = 8/3$

$$x_1(0) = 1.0$$
$$y_1(0) = 1.0$$
$$z_1(0) = 1.0$$

$$x_1(0) = 1.001$$
$$y_1(0) = 1.0$$
$$z_1(0) = 1.0$$



E. N. Lorenz, J. Atmos. Sci. **20**,130 (1963).

Synchronization of Chaos

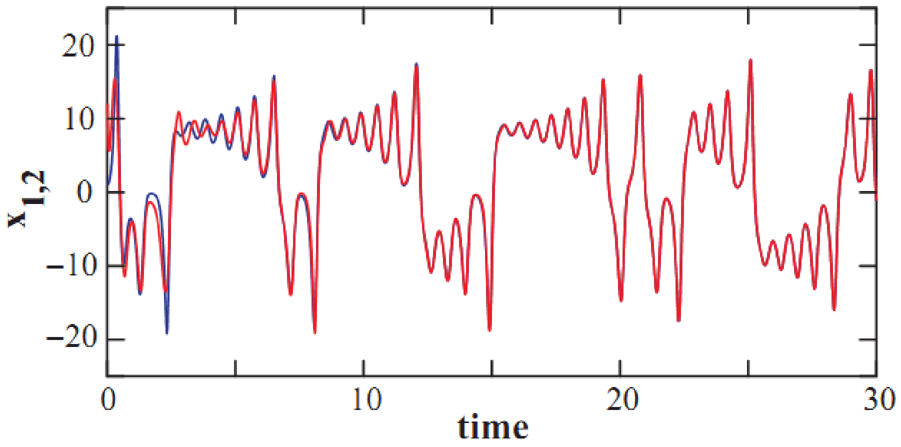
$$\frac{dx_2}{dt} = \sigma(y_2 - x_2) + 1.5(x_1 - x_2)$$

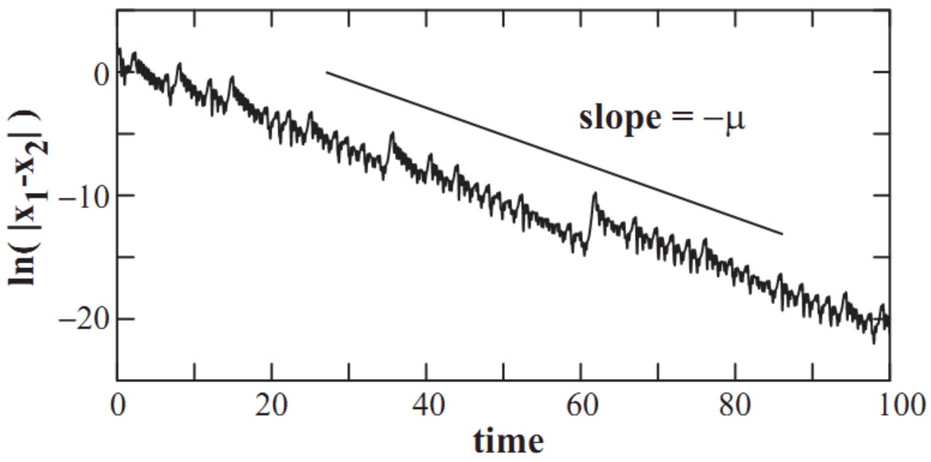
$$\frac{dy_2}{dt} = x_2(\rho - z_2) - y_2$$

$$\frac{dz_2}{dt} = x_2 y_2 - \beta z_2$$

$$x_1(0) = 1.0 \quad y_1(0) = 1.0 \quad z_1(0) = 1.0$$

$$x_2(0) = 12.0 \quad y_2(0) = 1.0 \quad z_2(0) = 5.0$$

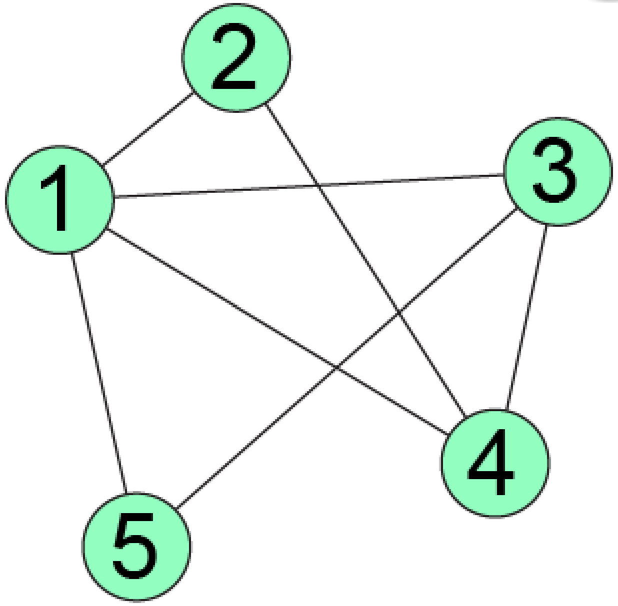




Reduction approach to analyze stability

- From a large system of coupled differential equations (high-dimensional) to a reduced set of equations(low-dimensional)
- We analyze local stability: why?
- Advantage: provides necessary and sufficient conditions
- Disadvantage: the approach relies on numerical calculations of the low-dimensional system
- However, under certain circumstances once the regions of stability for the low-dimensional systems are computed, they can be applied to several *types* of high-dimensional systems
- Typical example: Master Stability Functions
- Several types of limitations

Representing Networks and Graphs



$$C = \begin{bmatrix} \bullet & 1 & 1 & 1 & 1 \\ 1 & \bullet & 0 & 1 & 0 \\ 1 & 0 & \bullet & 1 & 1 \\ 1 & 1 & 1 & \bullet & 0 \\ 1 & 0 & 1 & 0 & \bullet \end{bmatrix}$$

- $C_{ij} = 1$, if node i and j are connected
- Assume all connections are identical, bidirectional
- Generalizations:
  - Weighted connections
  - Directional links ($C_{ij} \neq C_{ji}$)

Generalizations:
Weighted connections
Directional links ($C_{ij} \neq C_{ji}$)

Coupled Dynamical Systems

Continuous-time:

$$\frac{d}{dt}x_i(t) = F\big(x_i(t)\big) + \sum_{j=1}^{N} C_{ij} H\big(x_j(t)\big)$$

Discrete-time:

$$x_i[n+1] = F\big(x_i[n]\big) + \sum_{j=1}^{N} C_{ij} H(x_j[n])$$

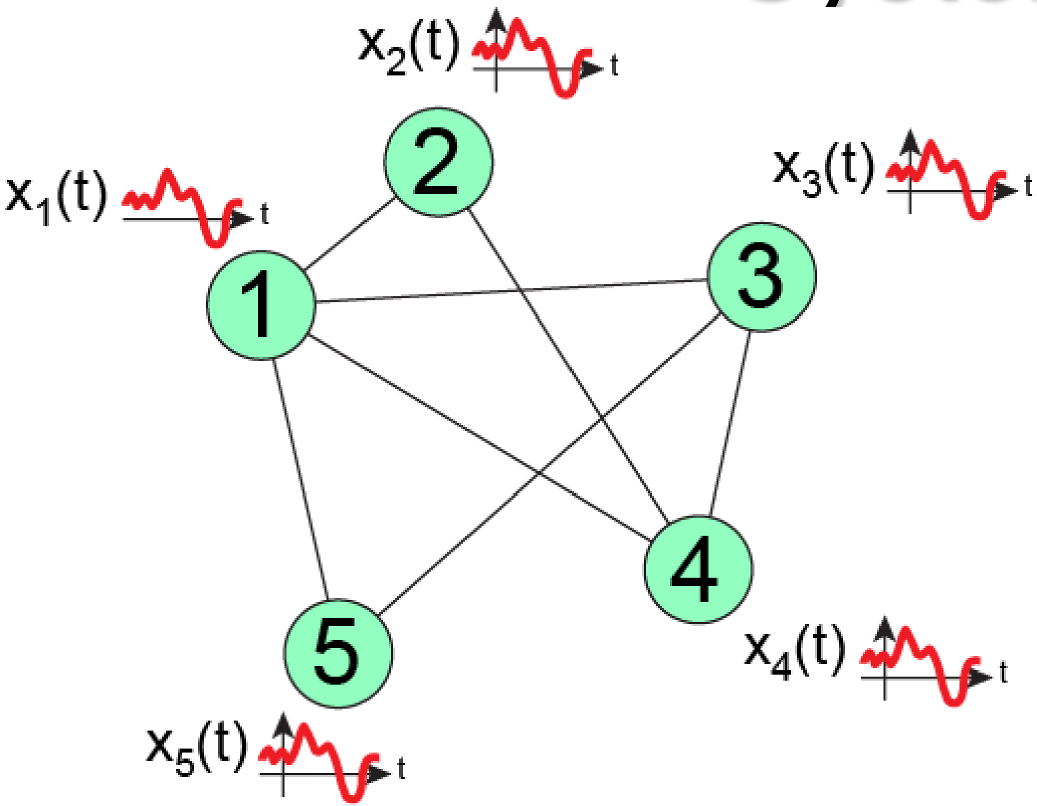
Q1: ***Can*** these equations synchronize?

(Do they *admit* a synchronous solution $x_1 = x_2 = \ldots x_N$?)

Q2: ***Do*** these equations synchronize?

(... and is the synchronous solution stable?)

Synchronization of Coupled Systems



Laplacian Coupling Matrix (row sum = 0):

$$C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -3 & 1 & 1 \\ 1 & 1 & 1 & -3 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{bmatrix}$$

$$x_1(t) = x_2(t) = \ldots \equiv x_s(t), \quad \frac{d}{dt} x_s(t) = F\big(x_s(t)\big)$$

Reduction approach to analyze stability

- From a large system of coupled differential equations (high-dimensional) to a reduced set of equations(low-dimensional)
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- Disadvantage: the approach relies on numerical calculations of the low-dimensional system
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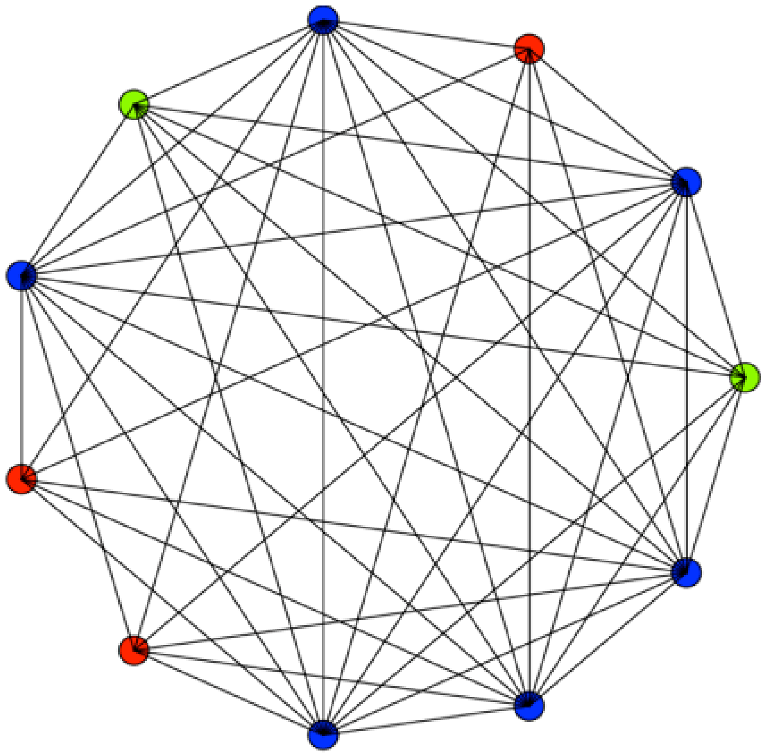
(Free) Tools for Computing Symmetries

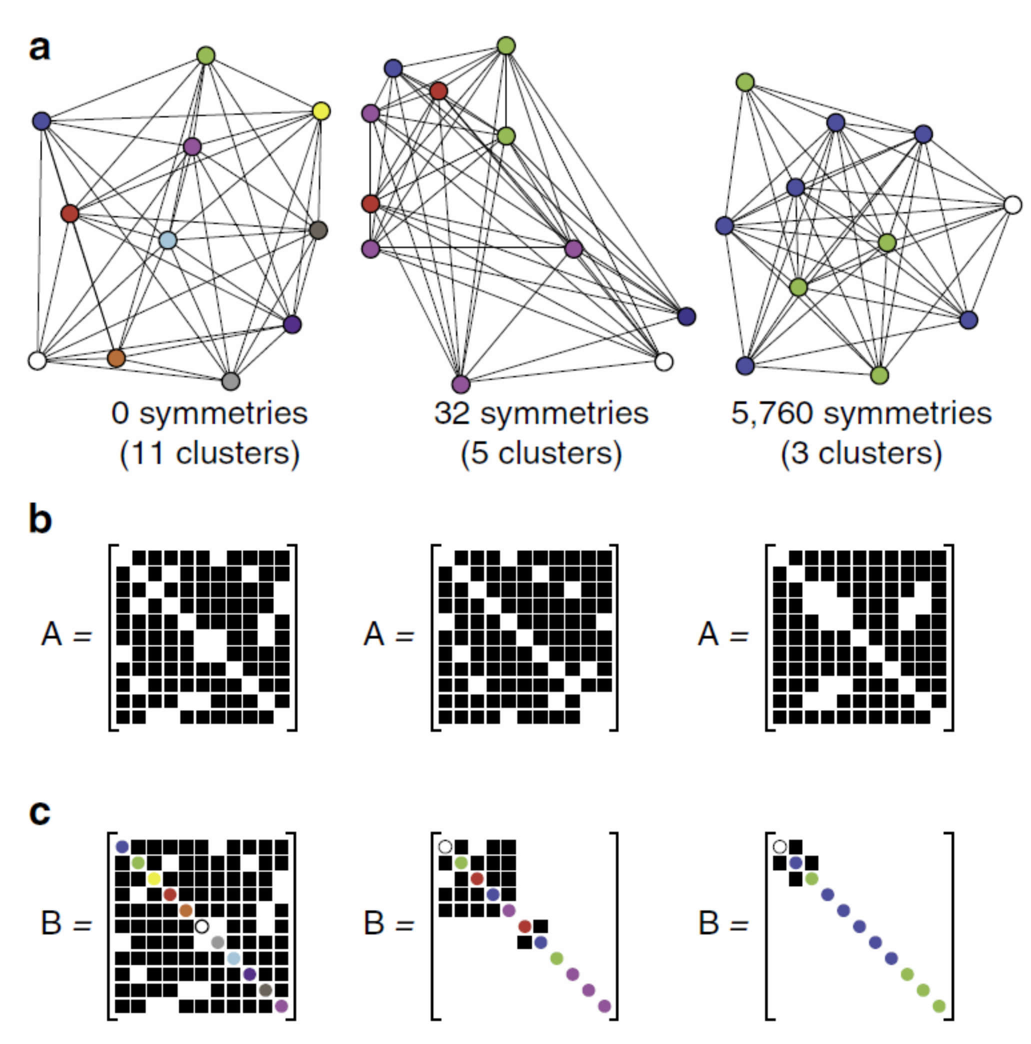
- **GAP** = Groups, Algorithms, Programming (software for computational discrete algebra) http://www.gap-system.org/
- **Sage** = Unified interface to 100's of open-source mathematical software packages, including GAP http://www.sagemath.org/
- **Python** = Open-source, multi-platform programming language http://www.python.org/

Example Output (GAP/Sage)

```
G.order(), G.gens()= 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
  Node 2

orb= [1, 5]
nodeSyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cycleSyncvec  [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
  Node 1

orb= [2, 4, 11, 6, 9, 10]
nodeSyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cycleSyncvec  [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]
  Node 4

orb= [3, 7, 8]
nodeSyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cycleSyncvec  [0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]
```



Cluster Synchronization



Three randomly generated networks with varying amounts of symmetry and associated coupling matrices.

Stability of Synchronization

linearizing about cluster states

- C = coupling matrix in “node” coordinate system
- T = unitary transformation matrix to convert to IRR coordinate system
- B = TCT^{-1} = block-diagonononalized form

Cluster Synchronization (contd.)

Variational equations about the synchronized solutions

$$\delta\dot{x}(t) = \left[\sum_{m=1}^{M} E^{(m)} \otimes DF\big(s_m(t)\big) + \sigma A \sum_{m=1}^{M} E^{(m)} \otimes DH\big(s_m(t)\big)\right]\delta x(t)$$

Where the Nn-dimensional vector $\delta x(t) = [\delta x_1(t)^T, \delta x_2(t)^T, \dots, \delta x_N(t)^T]^T$ and $E^{(m)}$ is an N-dimensional diagonal matrix such that

$$E_{ii}^{(m)} = \begin{cases} 1, & if \;\; i \in C_m \\ 0, & \text{otherwise} \end{cases}$$

And, C_m be the set of nodes in the m-th cluster with synchronous motion $s_m(t)$.

Define a transformed coupling matrix $B = TAT^{-1}$, where T is the transformation matrix.

Cluster Synchronization (contd.)

Applying T to variational equation we get

$$\dot{\eta}(t) = \left[\sum_{m=1}^{M} J^{(m)} \otimes DF\big(s_m(t)\big) + \sigma B \otimes I_N \sum_{m=1}^{M} J^{(m)} \otimes DH\big(s_m(t)\big) \right] \eta(t)$$

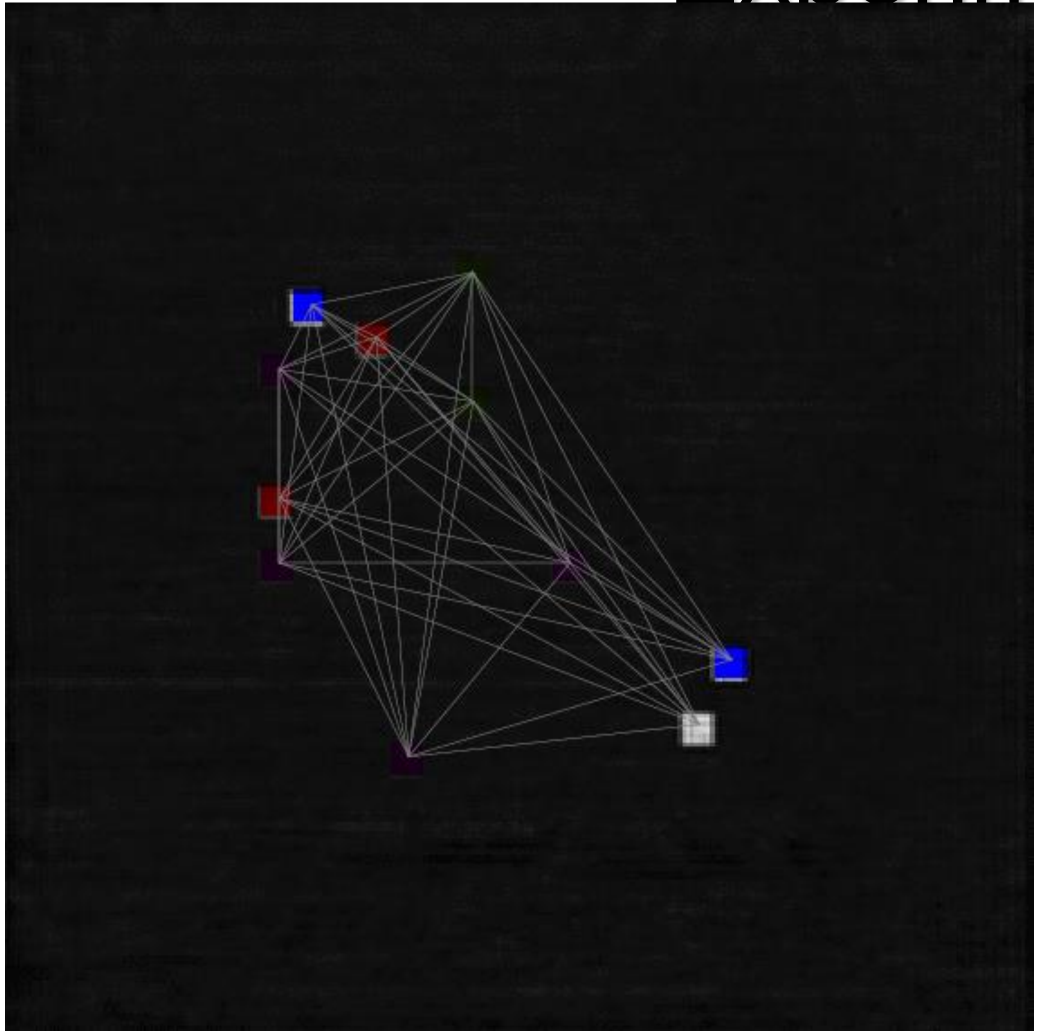
Where $\eta(t) = T \otimes \delta x(t)$ and $J^{(m)}$ is the transformed $E^{(m)}$.

We can write the block diagonal B as a direct sum,

$$B = \oplus_{l=1}^{L} I_{d^{(l)}} \otimes C^{l},$$

where C^l is a (generally complex) $p_l \times p_l$ matrix with p_l = the multiplicity of the lth IRR in the permutation representation $\{R_g\}$, L = the number of IRRs present and $d^{(l)}$ = the dimension of the lth IRR, so that $\sum_{l=1}^{L} d^{(l)} p_l = N$.

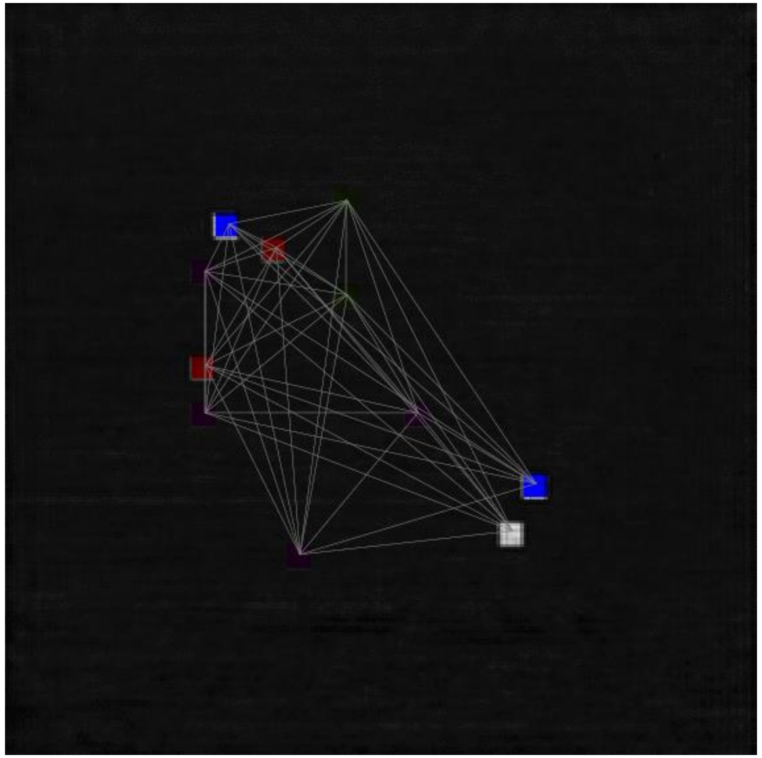
Cluster Synchronization in Experiment



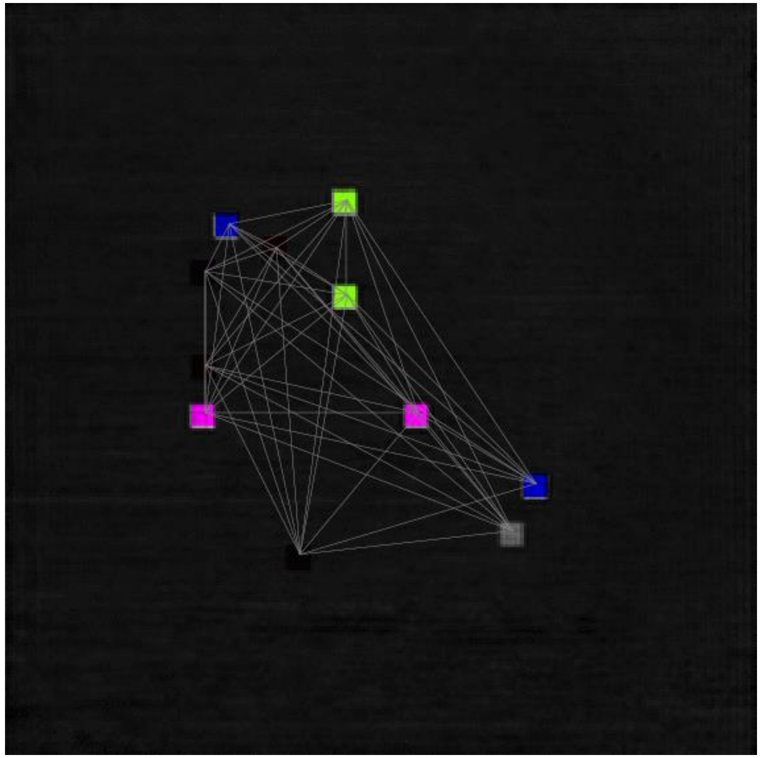
- 11 nodes
- 49 links
- 32 symmetries
- 5 clusters:
  - Blue (2)
  - Red (2)
  - Green (2)
  - Magenta (4)
  - White (1)

Isolated Desynchronization

- Pay attention to the magenta cluster:

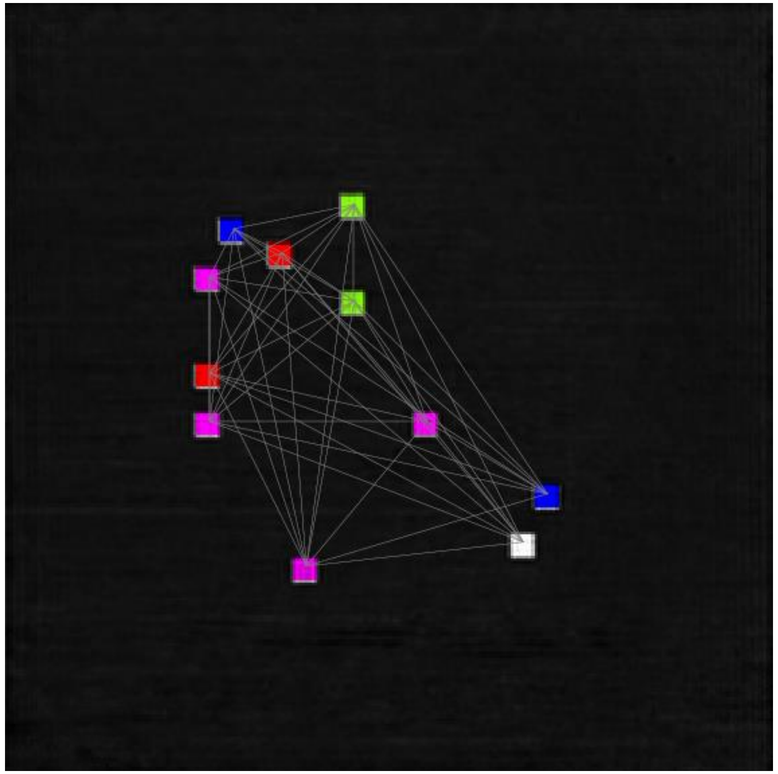


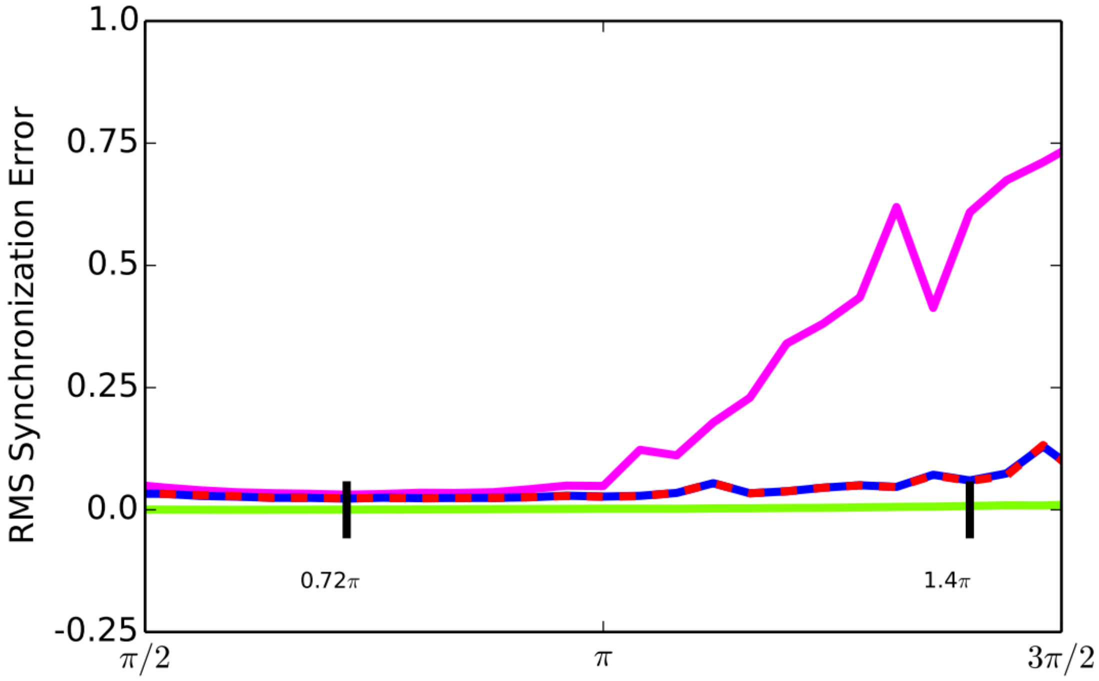
a = 0.7π



a = 1.4π

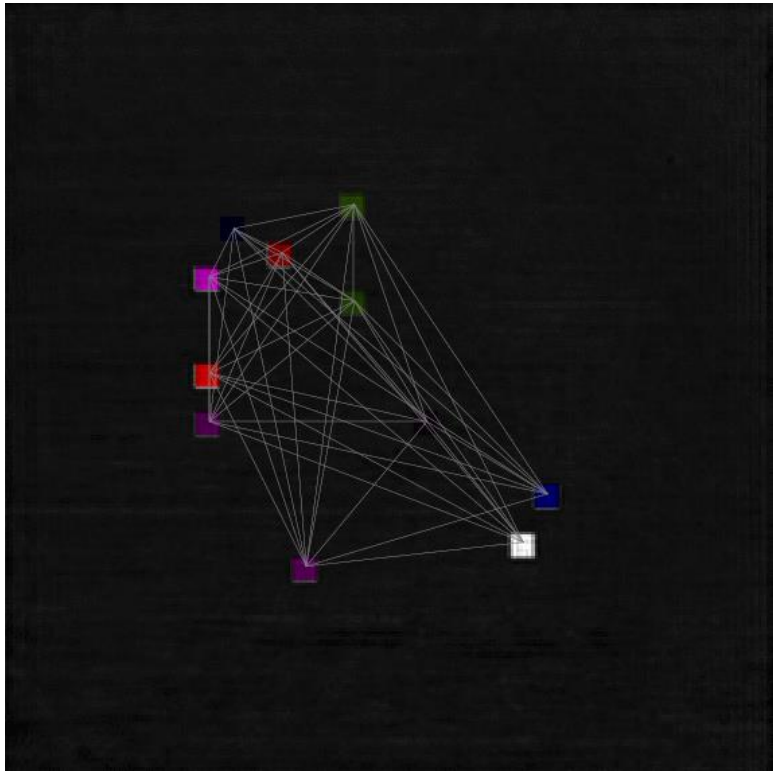
Synchronization Error

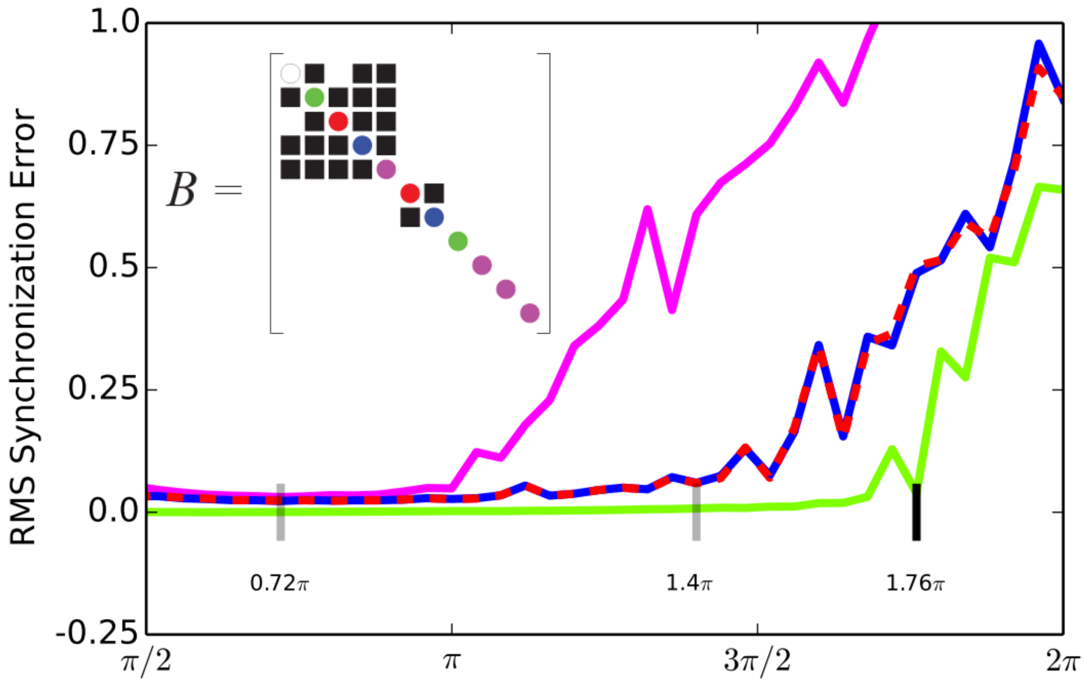




a

Intertwined Clusters



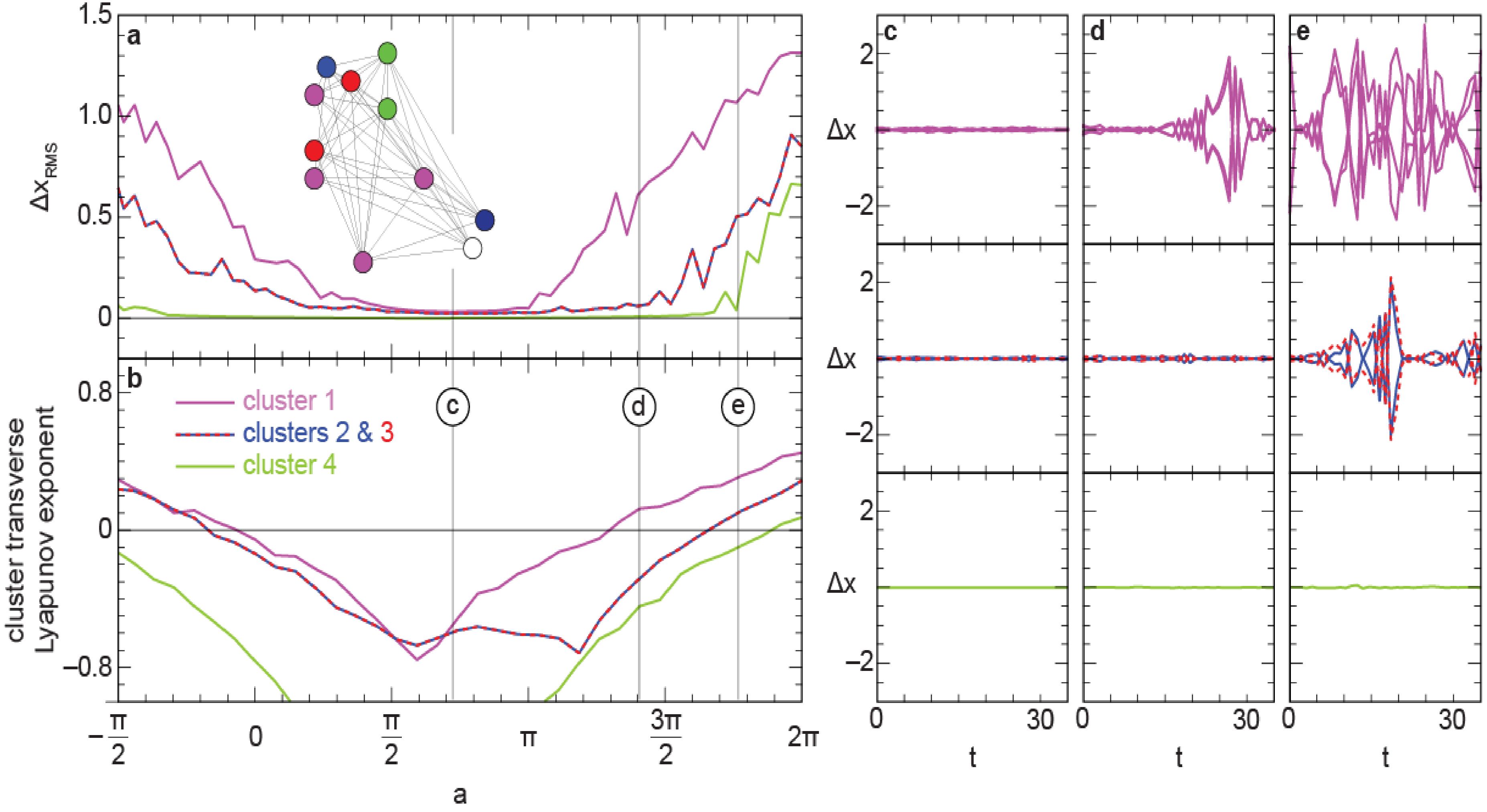


a

- Red and blue clusters are inter-dependent
- (sub-group decomposition)

Transverse Lyapunov Exponent

(linearizing about cluster synchrony)

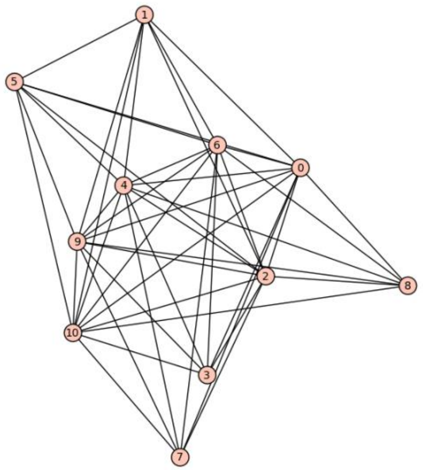


Symmetries and Clusters in Random Networks

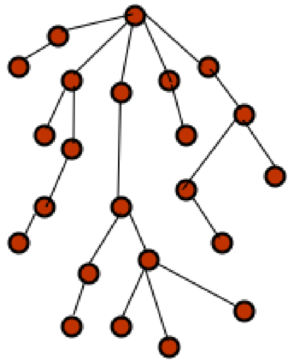
- N= 25 nodes (oscillators)
- 10,000 realizations of each type
- Calculate # of symmetries, clusters

Random

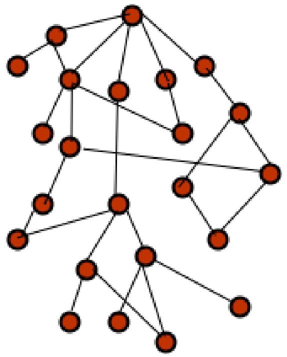
n_{delete}= 20



Scale-free Tree



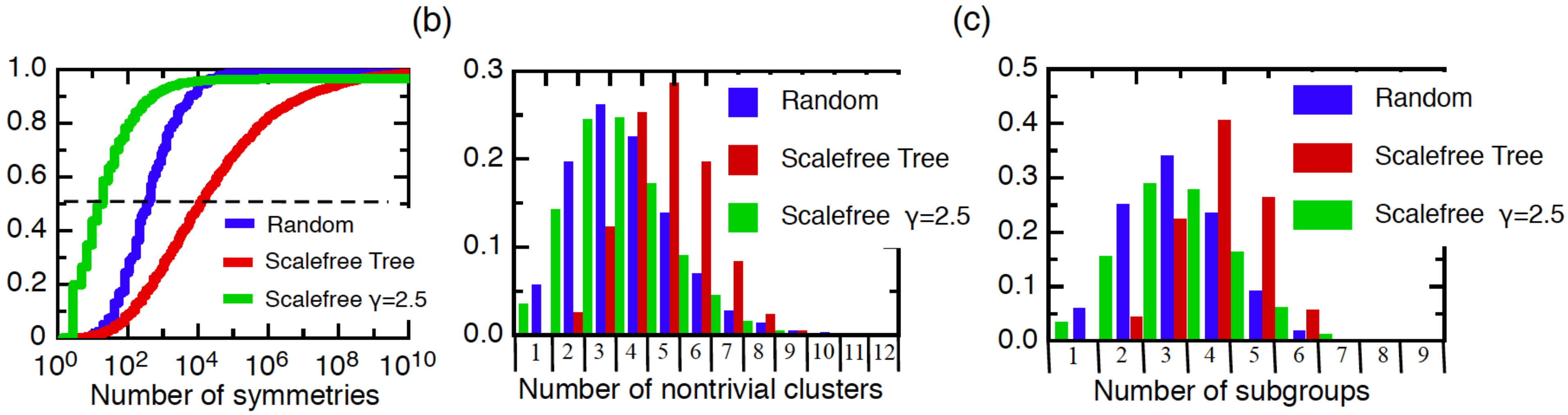
Scale-free γ



A.-L. Barabasi and R. Albert, “Emergence of scaling in random networks," *Science* **286**, 509-512 (1999).

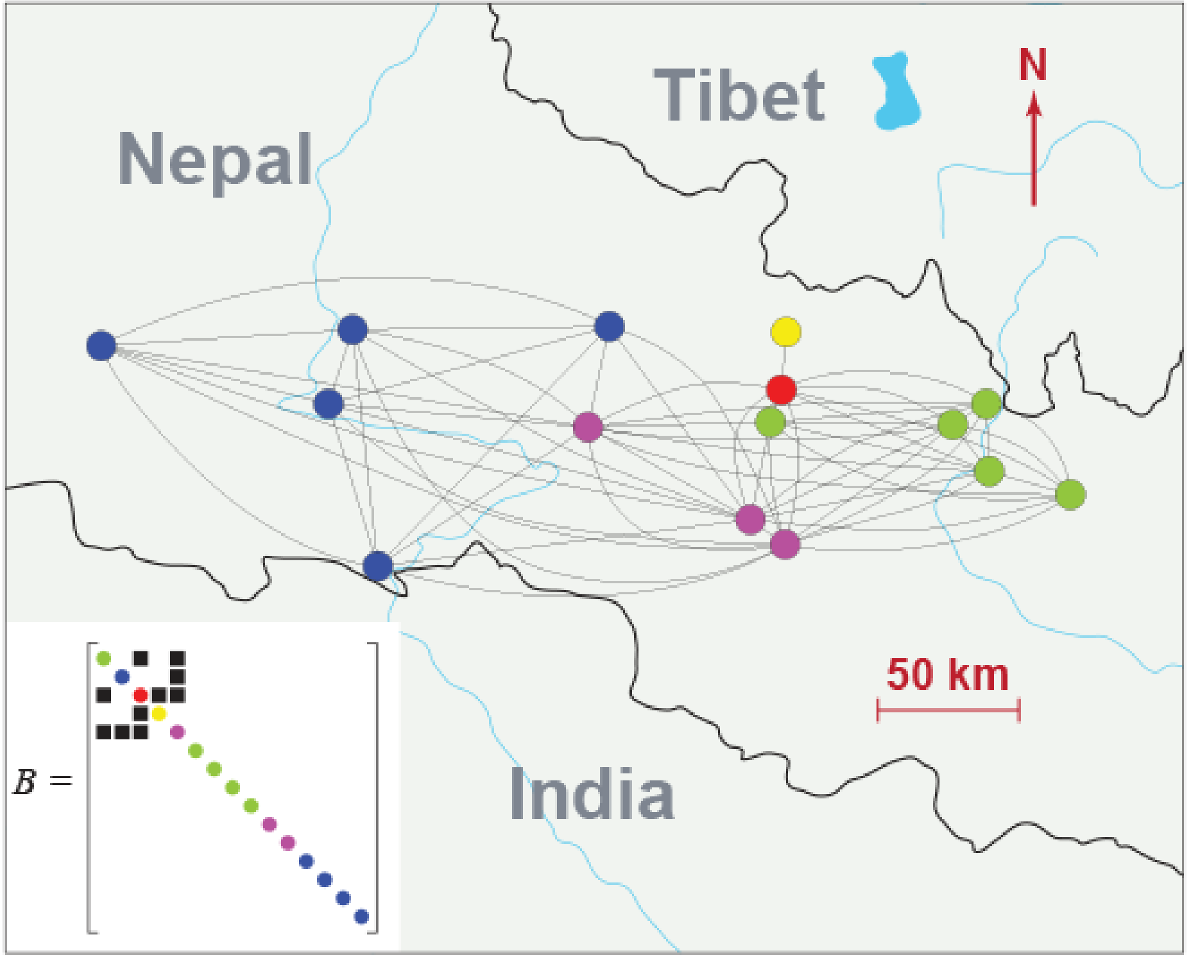
K-I Goh, B Kahng, and D Kim, “Universal behavior of load distribution in scale-free networks,“ *Phys. Rev. Lett.* **87**, 278701 (2001).

Symmetry Statistics

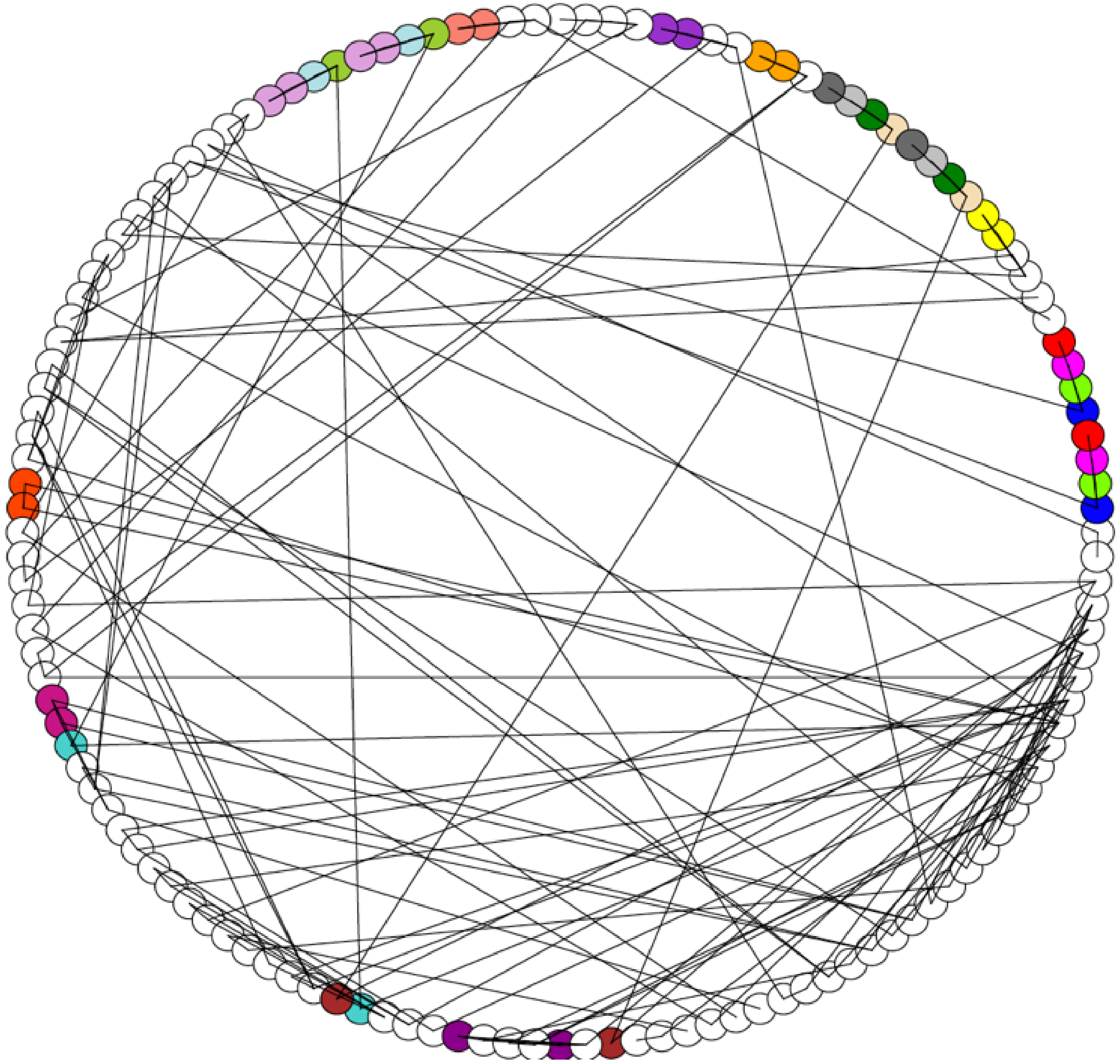


Symmetries, clusters and subgroup decompositions seem to be universal across many network models

Power Network of Nepal



Mesa Del Sol Electrical Network



- 4096 symmetries
- 132 Nodes
- 20 clusters
- 90 trivial clusters
- 10 subgroups

Symmetries & Clusters in Larger Networks

MacArthur *et al.*, “On automorphism groups of networks," *Discrete Appl. Math.* **156**, 3525 (2008).

Network	Number of Nodes $N_{\mathcal{G}}$	Number of Edges $M_{\mathcal{G}}$	Number of Symmetries $a_{\mathcal{G}}$
Human B Cell Genetic Interactions[3]	5,930	64,645	5.9374×10^{13}
C. elegans Genetic Interactions[26]	2,060	18,000	6.9985×10^{161}
BioGRID datasets[23]:			
Human	7,013	20,587	1.2607×10^{485}
S. cerevisiae	5,295	50,723	6.8622×10^{64}
Drosophila	7,371	25,043	3.0687×10^{493}
Mus musculus	209	393	5.3481×10^{125}
Internet (Autonomous Systems Level)[12]	22,332	45,392	$1.2822 \times 10^{11,298}$
US Power Grid[25]	4,941	6,594	5.1851×10^{152}

> 88% of nodes are in clusters in all above networks

Case of Laplacian coupling

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_j C_{ij} \mathbf{H}(\mathbf{x}_j)$$

The coupling matrix is Laplacian $\sum_{j=1}^{N} Cij = 0$

The equations allow a fully synchronous solution:
$x_1 = x_2 = \ldots = x_N$

Other solutions may emerge where nodes are synchronized in clusters

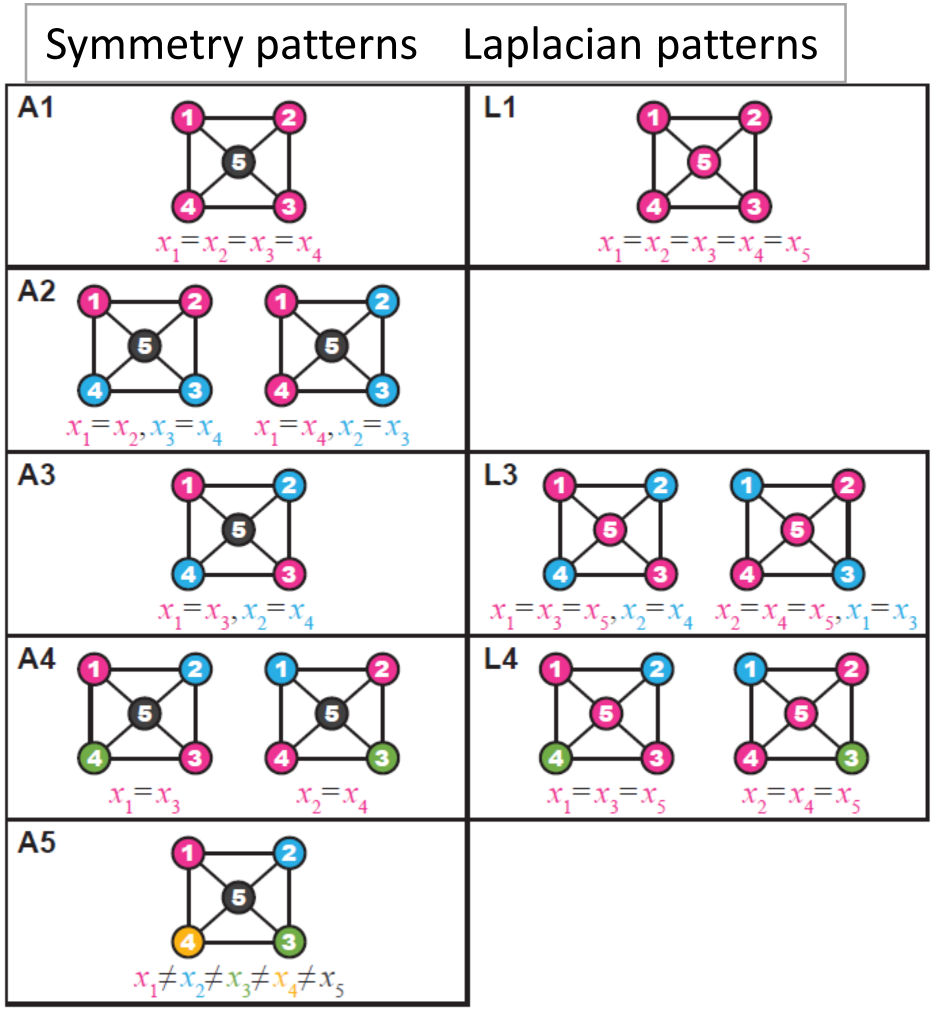
How can we find them?
How can we study stability?

An example:
a simple 5-node network

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

The figure on the right shows all of the synchronization patterns that may emerge

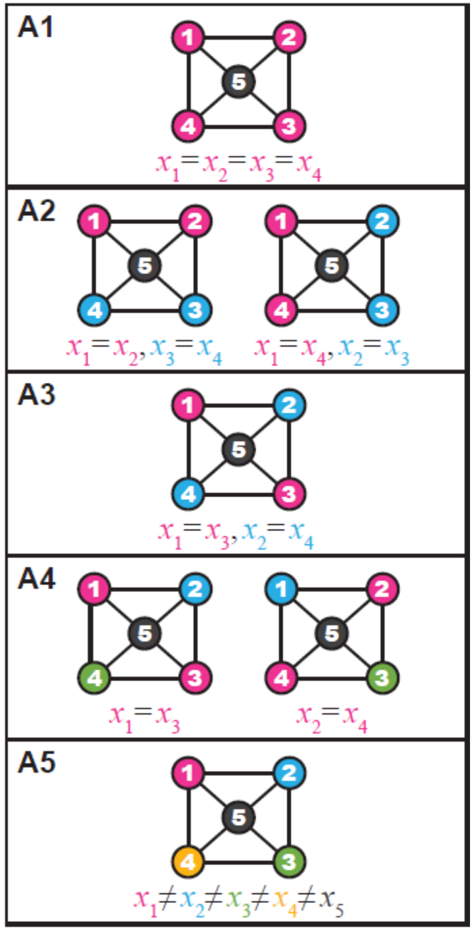


The A patterns

The *A* patterns are those that would be possible if the coupling matrix was the adjacency matrix

They can be obtained by progressively breaking up the maximal symmetry pattern (A1) up to the trivial pattern (A5)

They can be found by using computational group theory routines that generate all of the subgroups of the automorphism group

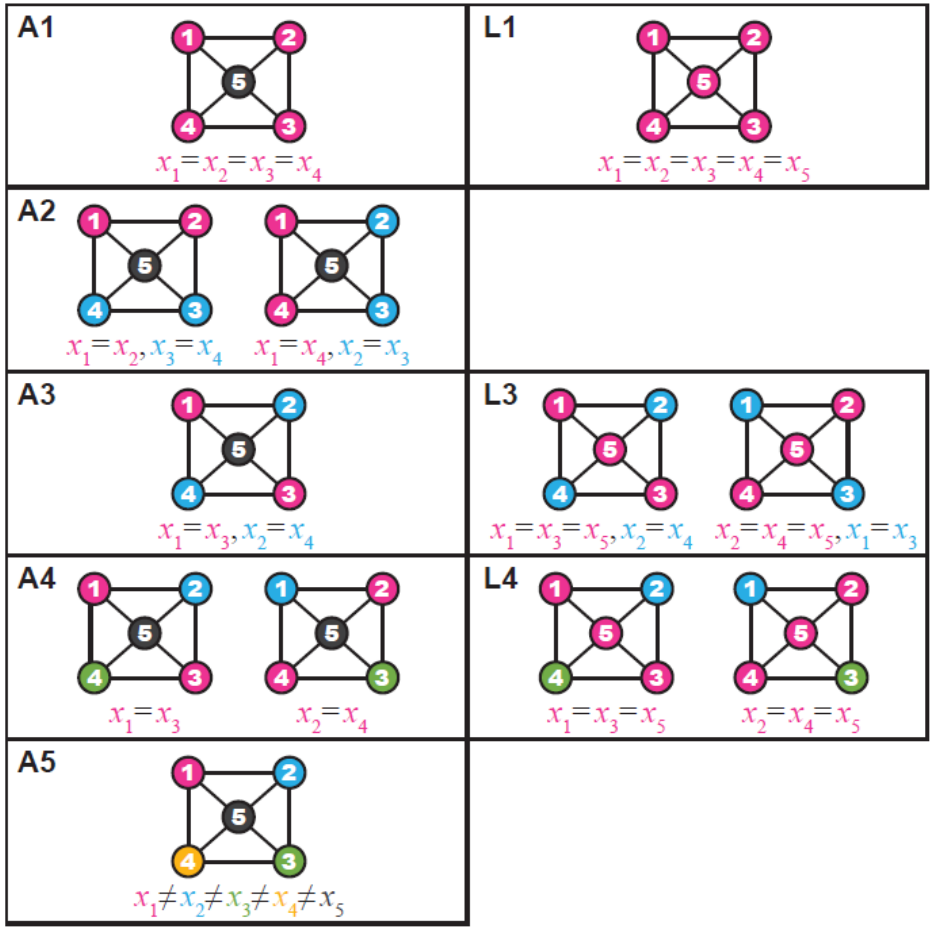


The L patterns

For each A pattern, several L patterns can be generated by merging the nodes in different clusters (merging not allowed by the symmetries) – a test is needed

In this particular example, the test reduces to checking whether node 5 can be included in an other existing cluster

Symmetry patterns   Laplacian patterns



Cluster merges

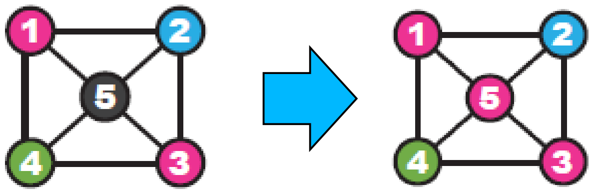
1) Observation: when clusters are synchronized, the diagonal feedback terms for each node cancel with the coupling terms from nodes in the same cluster.

2) To test for a merge, remove the inter-cluster couplings and adjust the diagonal entries accordingly → obtain a new coupling matrix

3) Compute the subgroups of the new coupling matrix.

4) If any of these subgroups includes nodes originally belonging to different clusters, then their dynamics is flow-invariant in the synchronized state and the cluster merging is possible.

Example: Pattern A4



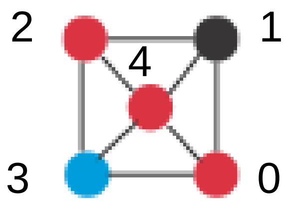
$$L = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix} \Rightarrow L_{eq} = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}$$

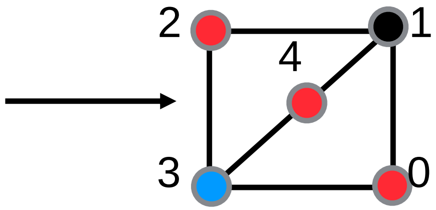
Check to see if dynamics allow the new synchronized cluster

Original Laplacian

$$\begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

L





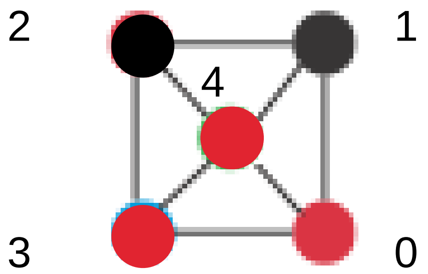
Dynamically equivalent Laplacian

$$\begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}$$

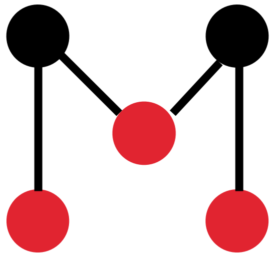
L_{eq}

[0,2,4] is a symmetry cluster of L_{eq}

Counter example: Not all combined clusters will work.



Combine clusters from different original clusters



dynamically equivalent network

Observations: beyond synchronization

Our routine, based on computational group theory, provides all of the possible patterns that can be observed for a given network topology

The routine's input is the network coupling matrix (either adjacency or Laplacian) and its output is a list of all the patterns that can emerge – *some patterns are allowed, while others are not!*

We are able to answer these questions: *is this pattern at all possible for this network*? *What are (all) the patterns that are compatible with a given network?*

Our work has immediate practical relevance: for example to technological networks, neuronal networks, genetic networks, for which certain patterns may be ***good*** and others may be ***bad***.

Up to this point, our approach is purely topological, not dynamical.

The role of dynamics: stability

In Pecora & Carroll "Master stability functions for synchronized coupled systems", PRL (1998), the stability problem for the fully synchronous pattern (L1) was studied

In Pecora, Sorrentino, Hagerstrom, Murphy, and Roy, "Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries", Nature Communications (2014), we studied the stability problem for the *maximal symmetry pattern* (A1)

We are currently studying stability of the remaining patterns in a low-dimensional form

Q1: Which ones of the topologically valid patterns are stable?

Q2: Can we reduce the dimensionality of these stability problems?

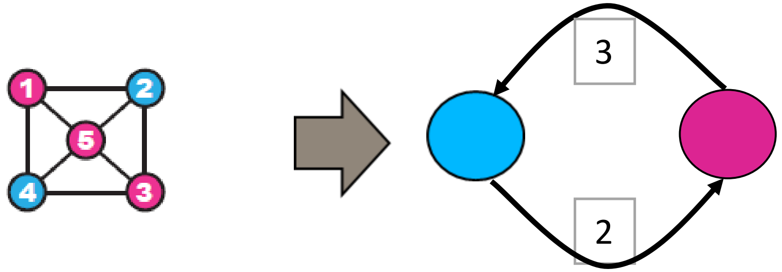
Quotient networks

To each pattern corresponds a quotient network

In the quotient network, each cluster is represented by only one node

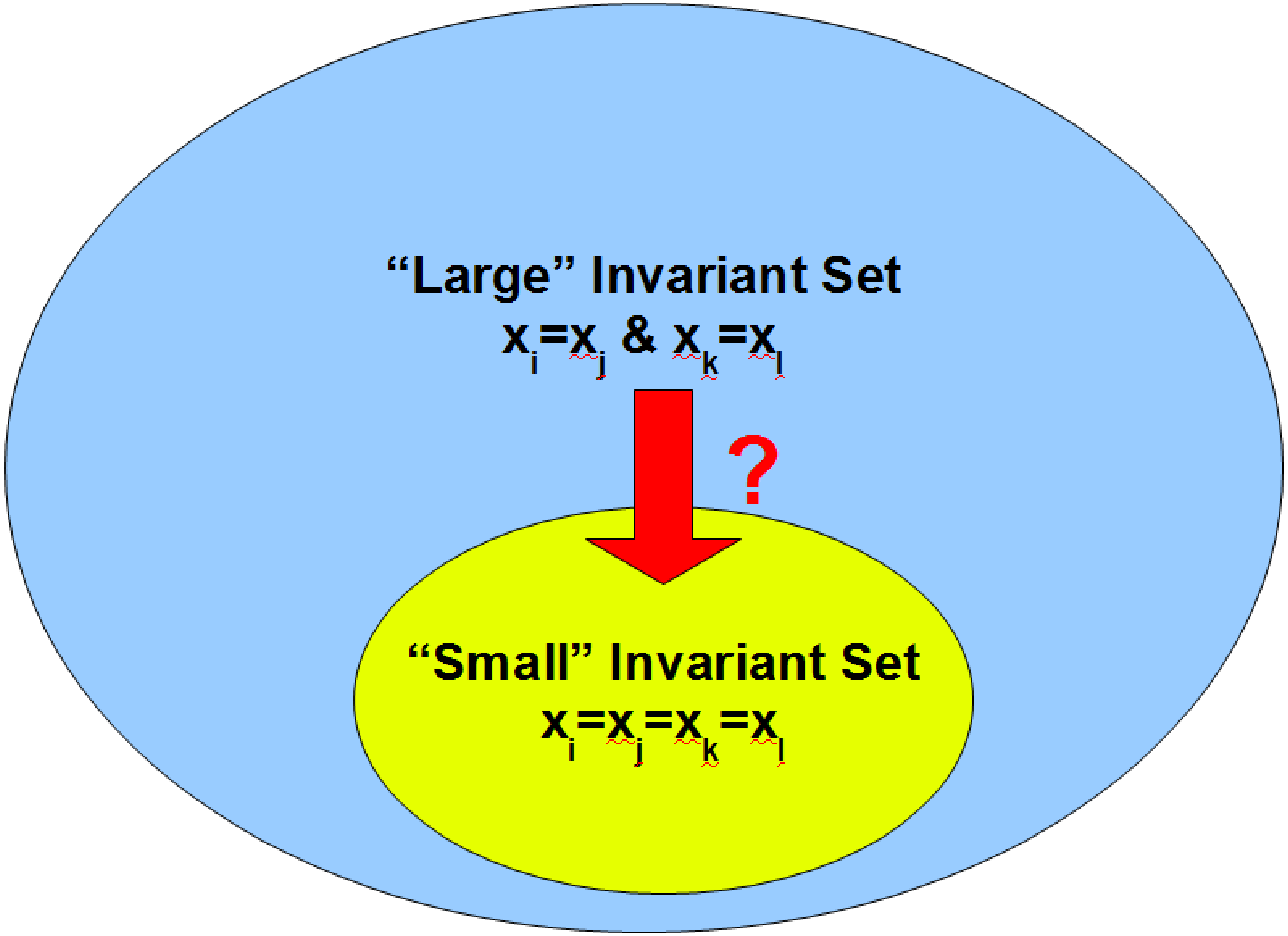
This is helpful because we can linearize about the quotient network dynamics and study stability

Example:

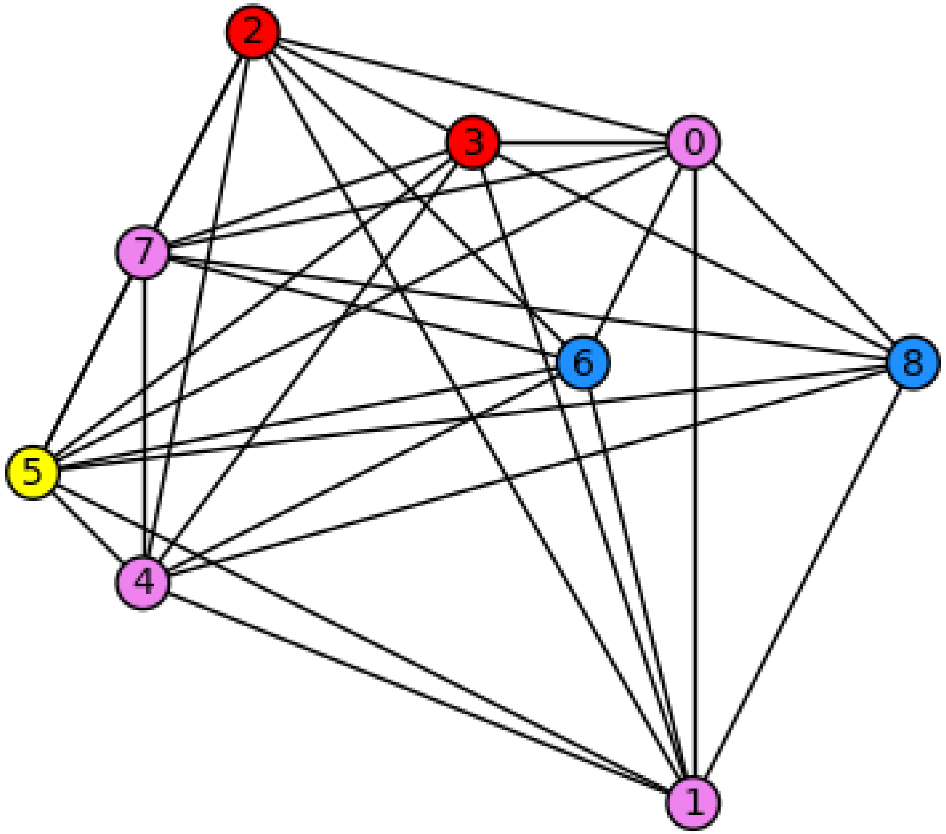


PROBLEM: not all the topologically valid patterns are also dynamically valid.
For example, under Laplacian coupling, it is possible that after a transient the blue and magenta clusters of the quotient network synchronize on the same time evolution.

NESTED INVARIANT SETS



Example: a 9-node “random” network



ORBITS:

Nodes 0,1,4,7. In what follows: 1,2,5,8.

Nodes 2,3. In what follows: 3,4

Nodes 6,8. In what follows: 7,9

RED AND BLUE CLUSTERS ARE INTERTWINED

Coupled map equations: Laplacian coupling

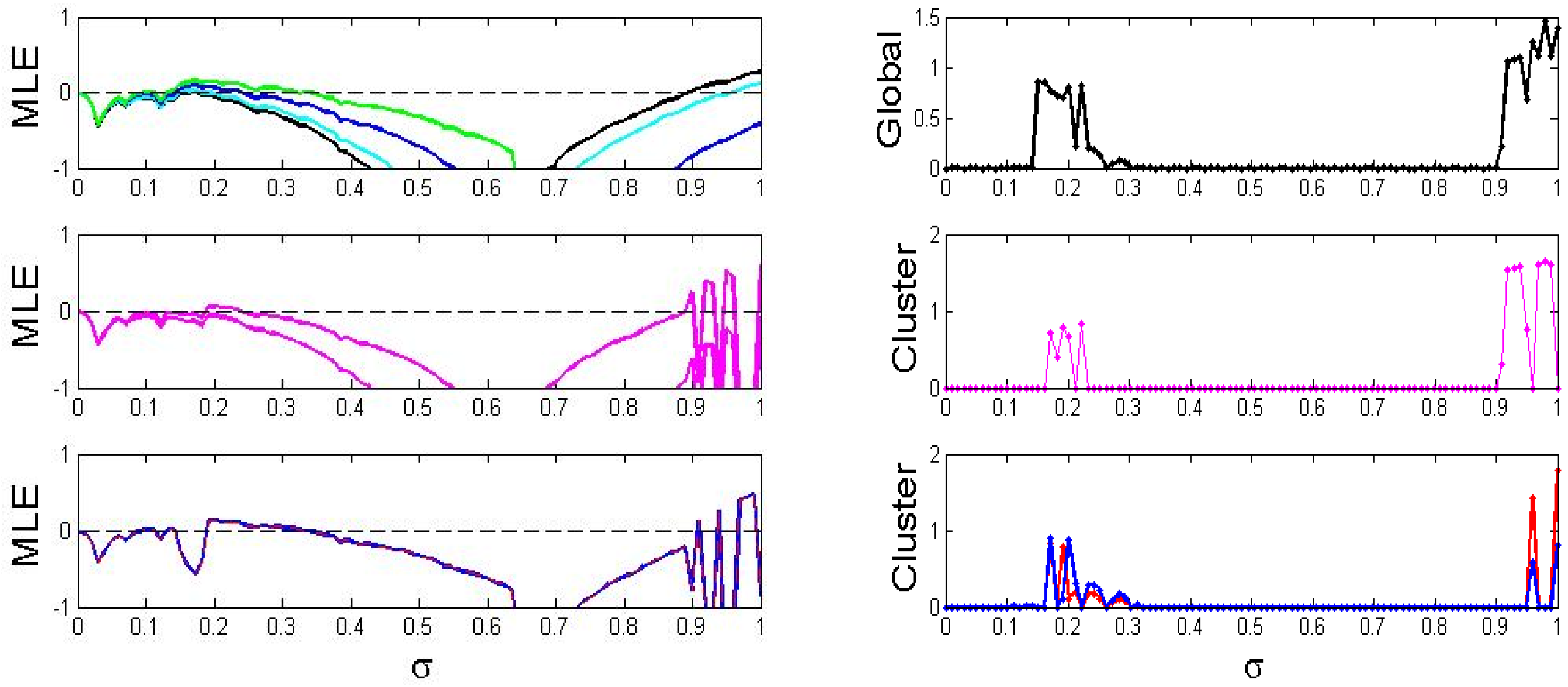
$$x_i[n+1] = \beta I(x_i[n]) + \sigma \sum_{j=1}^{N} C_{ij} I(x_j[n]) + \delta$$

$$\beta = 5$$

C is the Laplacian matrix corresponding to the N=9 node network in the previous slide, hence the fully synchronous pattern is a solution

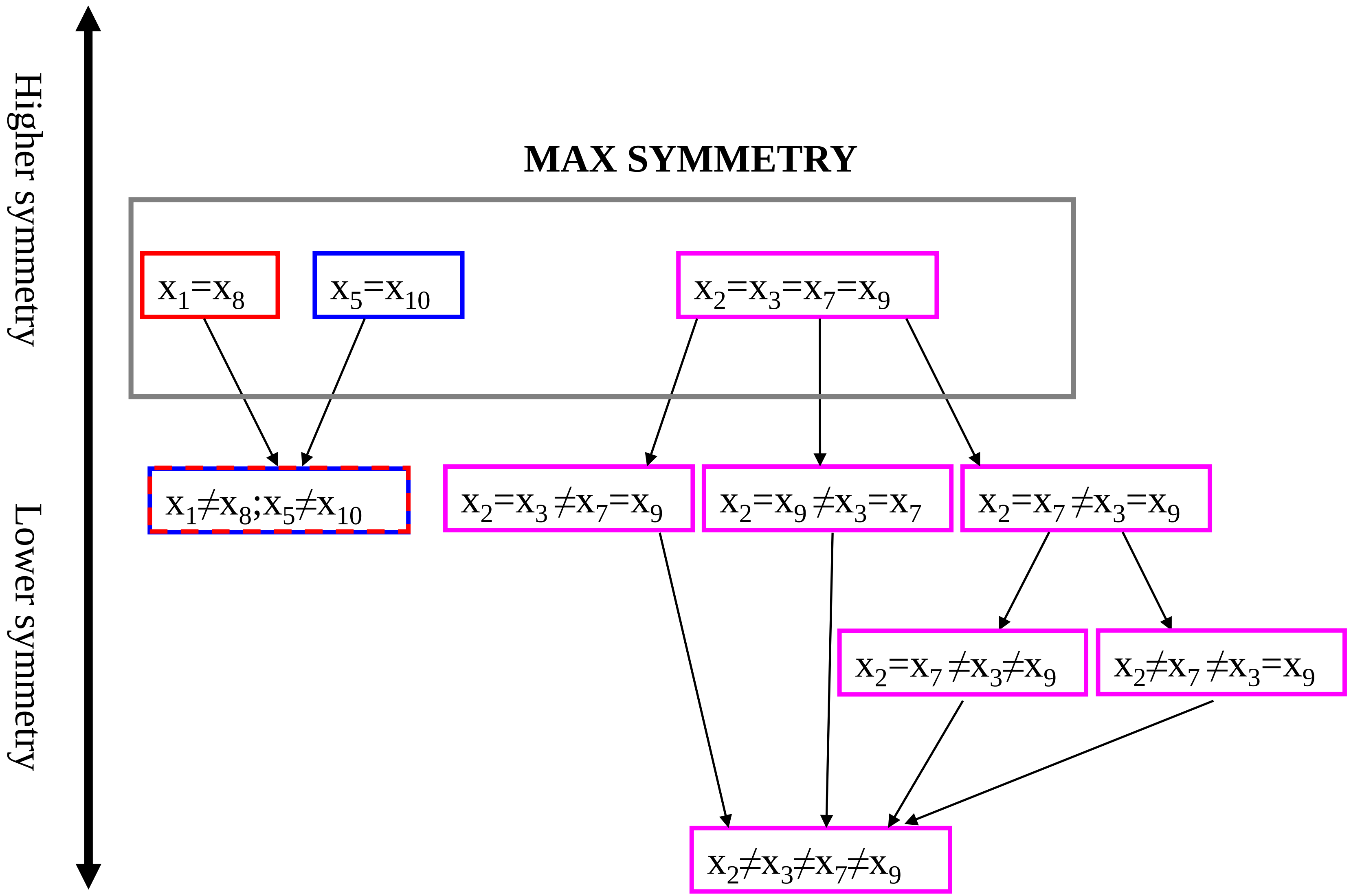
We swipe σ while keeping β fixed.

Stability of the fully sync state versus stability of the max symmetry solution

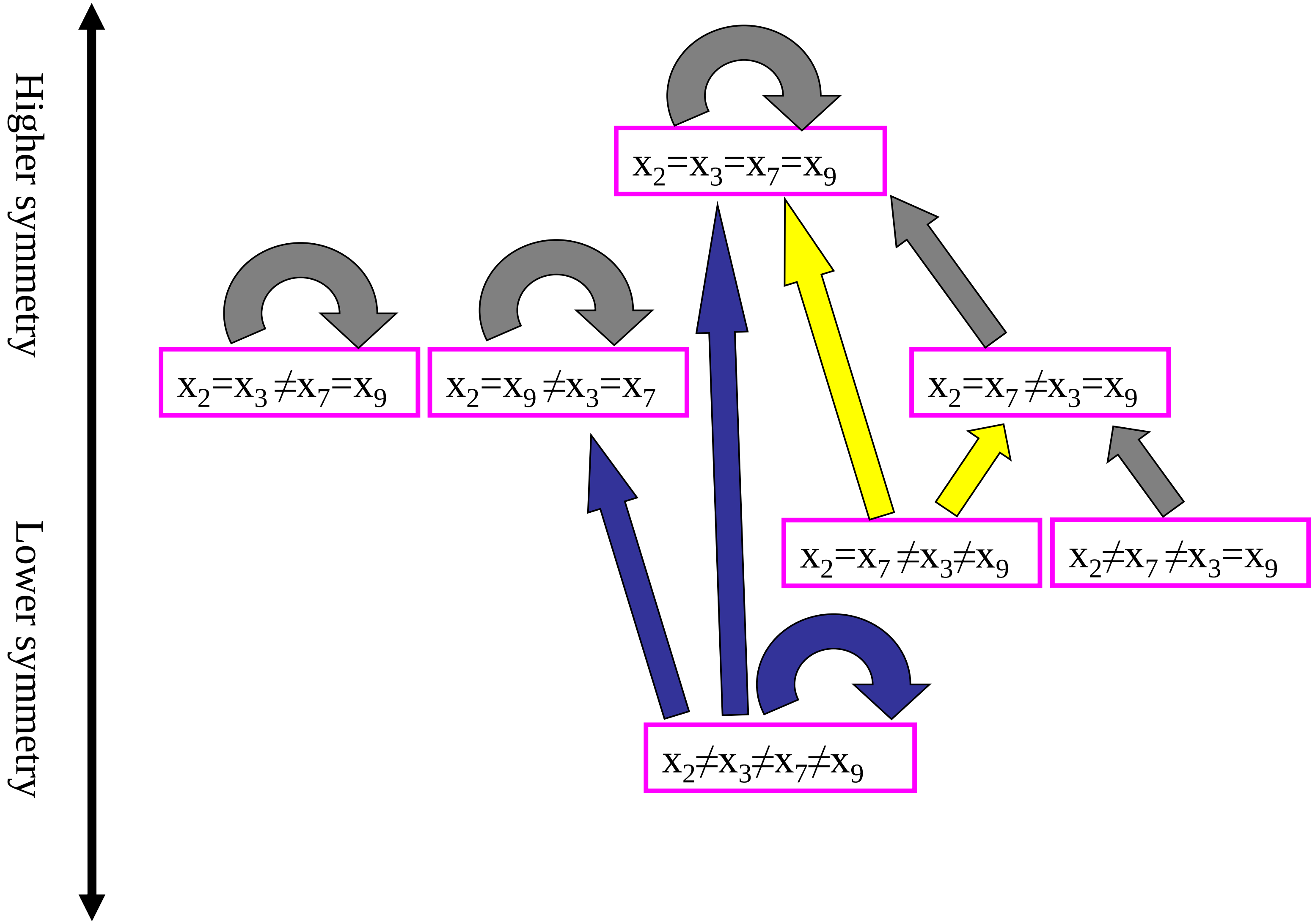


The figure on the right is for initial conditions close to the fully sync solution

Hierarchy of the symmetry patterns



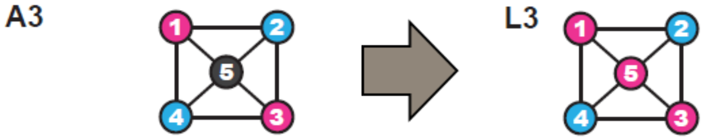
Dynamically valid patterns

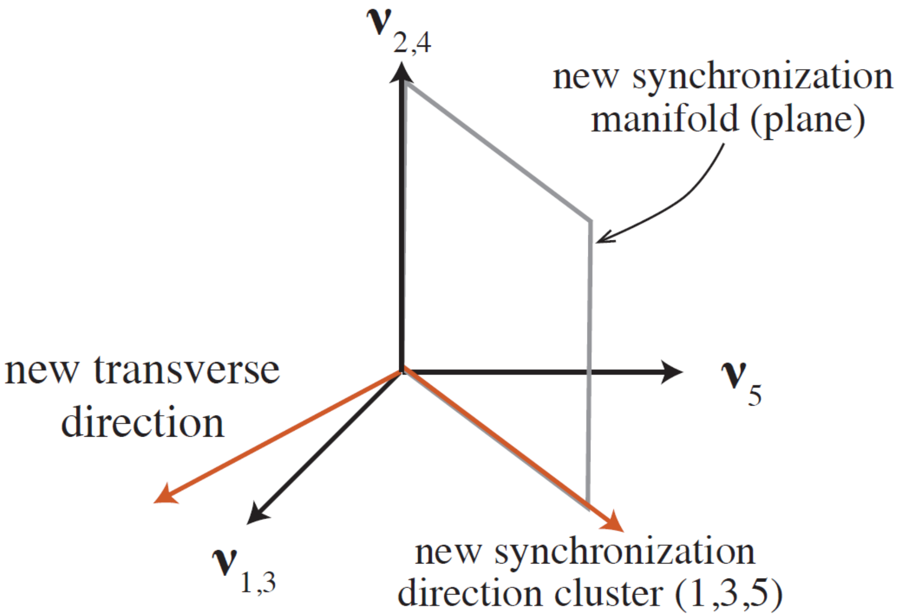


Stability of the lower-symmetry patterns

The block-diagonalized form for a given symmetry pattern can be obtained from that of another symmetry pattern

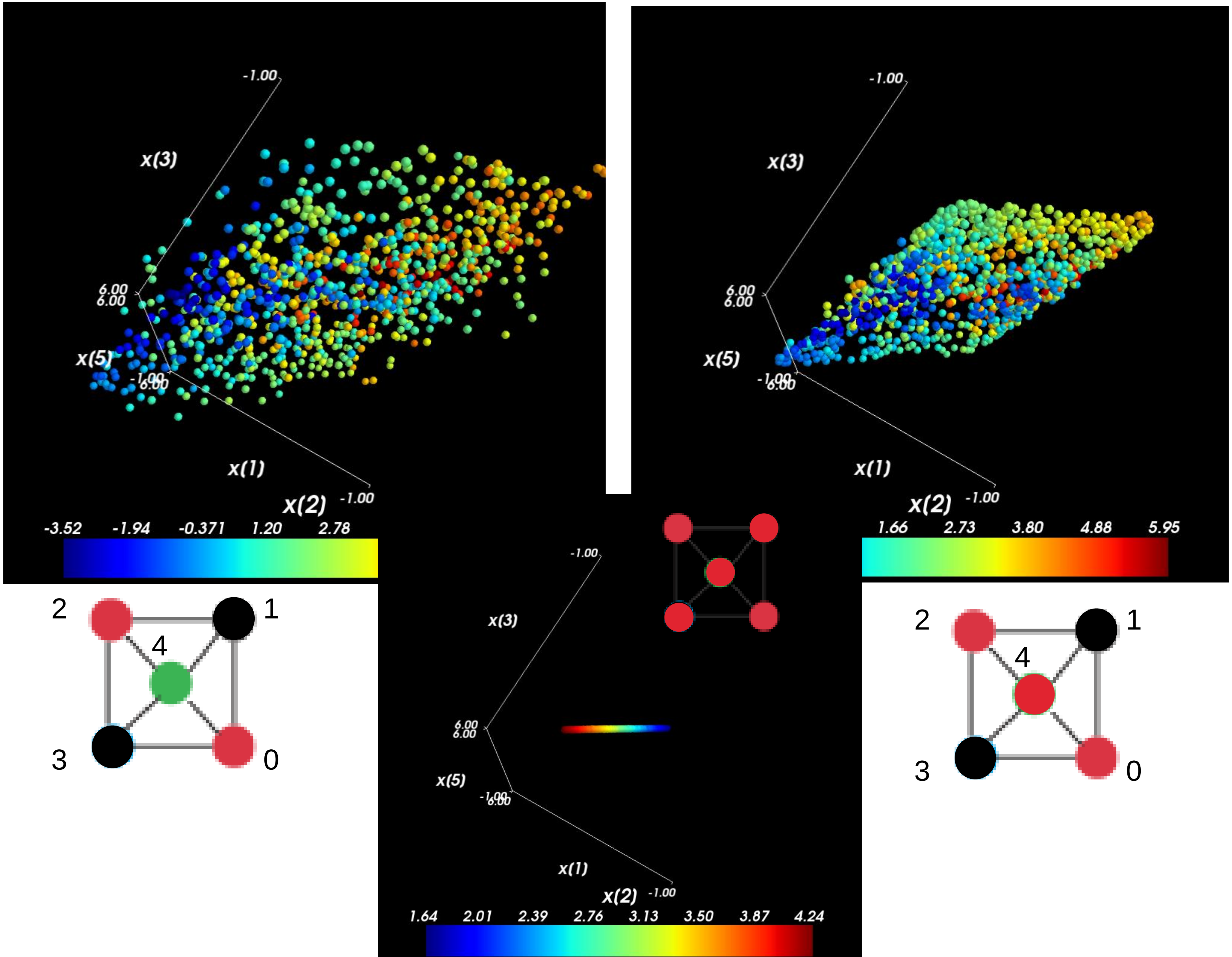
EXAMPLE: (A3 →L3)





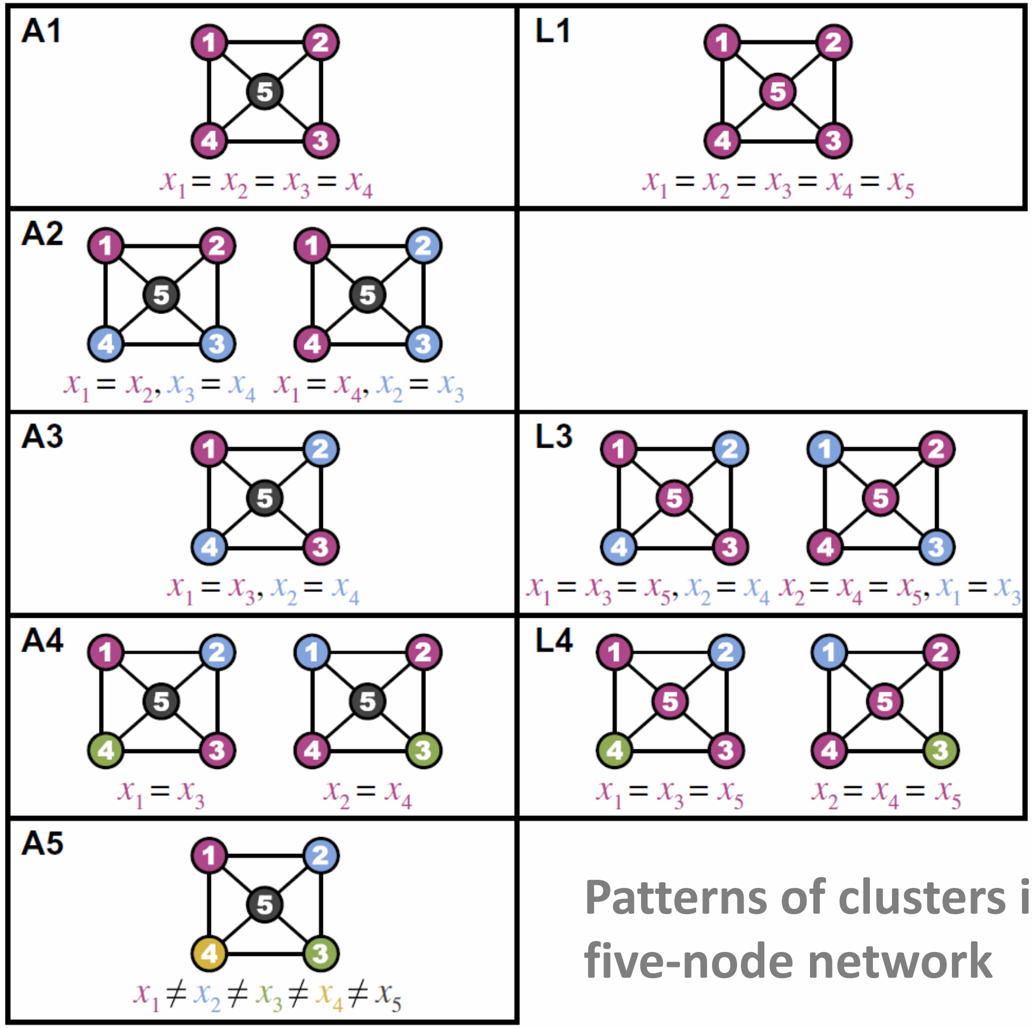
$$L' = \begin{pmatrix} -4.00 & -1.41 & 1.41 & 0.0 & 0.0 \\ -1.41 & -3.00 & -2.00 & 0.0 & 0.0 \\ 1.41 & -2.00 & -3.00 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -3.00 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3.00 \end{pmatrix}$$

$$L'' = \begin{pmatrix} -3.00 & 2.45 & 0.0 & 0.0 & 0.0 \\ 2.45 & -2.00 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -5.00 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -3.00 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3.00 \end{pmatrix}$$



-1.00
x(3)
6.00
6.00
x(5)
-1.00
6.00
x(1)
x(2)
-1.00
-3.52
-1.94
-0.371
1.20
2.78
1.66
2.73
3.80
4.88
5.95
1.64
2.01
2.39
2.76
3.13
3.50
3.87
4.24
2
1
4
3
0

Analyzing CS Pattern



Patterns of clusters in a five-node network

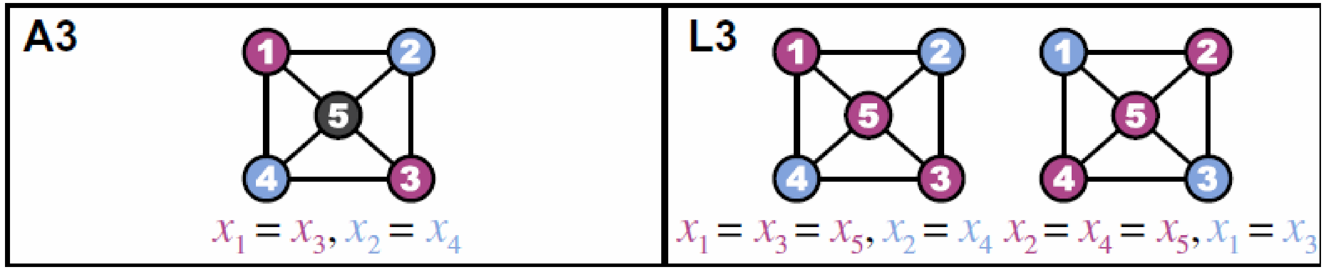
$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

Stability Analysis

Variational equation of the system equation,

$$\delta\dot{x}(t) = \left[\sum_{m=1}^{M} E^{(m)} \otimes DF\big(s_m(t)\big) + \sigma \sum_{m=1}^{M} \big(LE^{(m)}\big) \otimes DH\big(s_m(t)\big)\right] \delta x(t)$$



$$L' = \begin{pmatrix} -4 & -\sqrt{2} & \sqrt{2} & 0 & 0 \\ -\sqrt{2} & -3 & -2 & 0 & 0 \\ \sqrt{2} & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

Stability Analysis (Contd.)

Final variational equation for the L3 case,

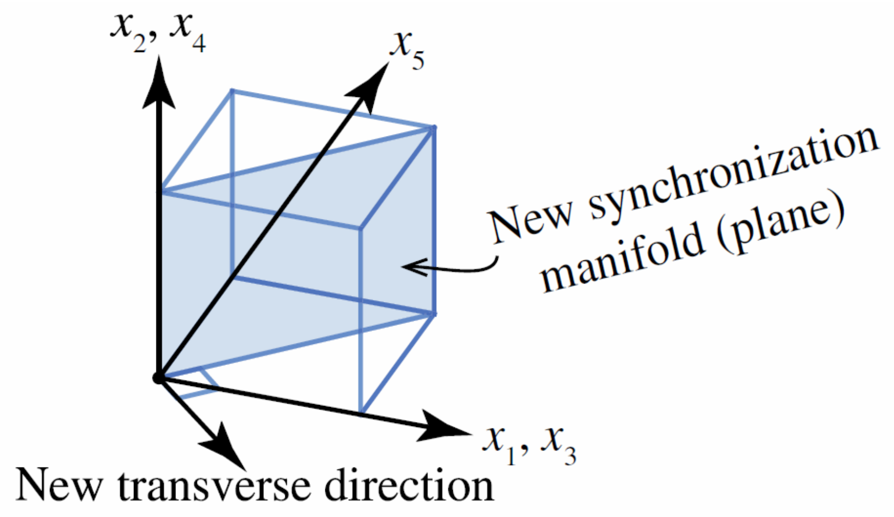
$$\dot{\eta} = \sum_{m=1}^{M} \left[J^{(m)} \otimes DF(s_m) + \sigma L'' J^{(m)} \otimes DH(s_m) \right] \eta$$

$$L'' = \begin{pmatrix} -3 & \sqrt{6} & 0 & 0 & 0 \\ \sqrt{6} & -2 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

Stability Analysis (Contd.)

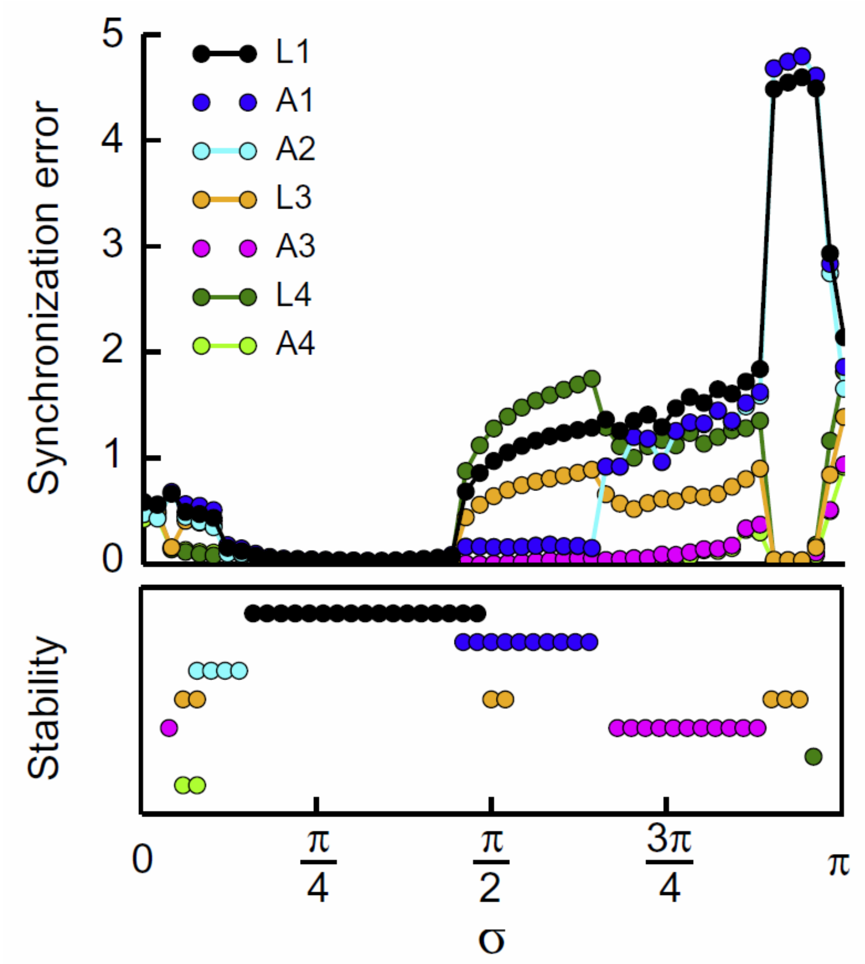
The dynamics of the system is modeled by a map according to,

$$x_i^{t+1} = \left[\beta \mathcal{I}\left(x_i^t\right) + \sigma \sum\nolimits_j L_{ij}\, \mathcal{I}\left(x_i^t\right) + \delta \right] \mathrm{mod}\ 2\pi$$



Reduction of the dimension of the three-dimensional synchronization manifold.

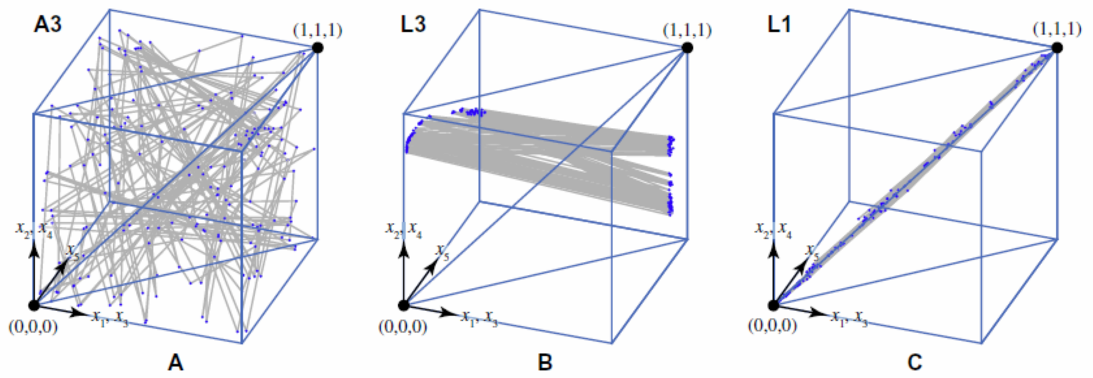
Stability Analysis (Contd.)



Experimental synchronization error for each synchronization pattern as a function of the parameter s for a five-node experimental system modeled

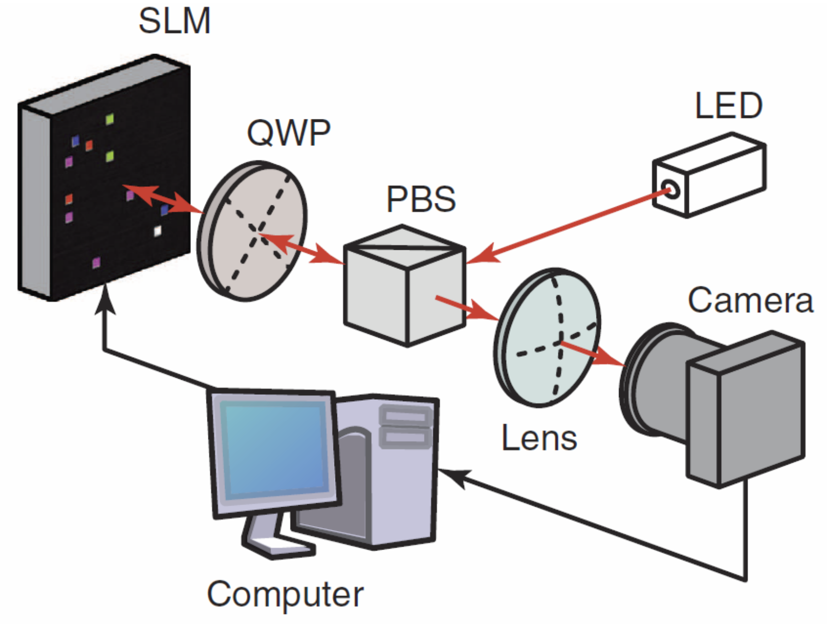
Stability Analysis (Contd.)

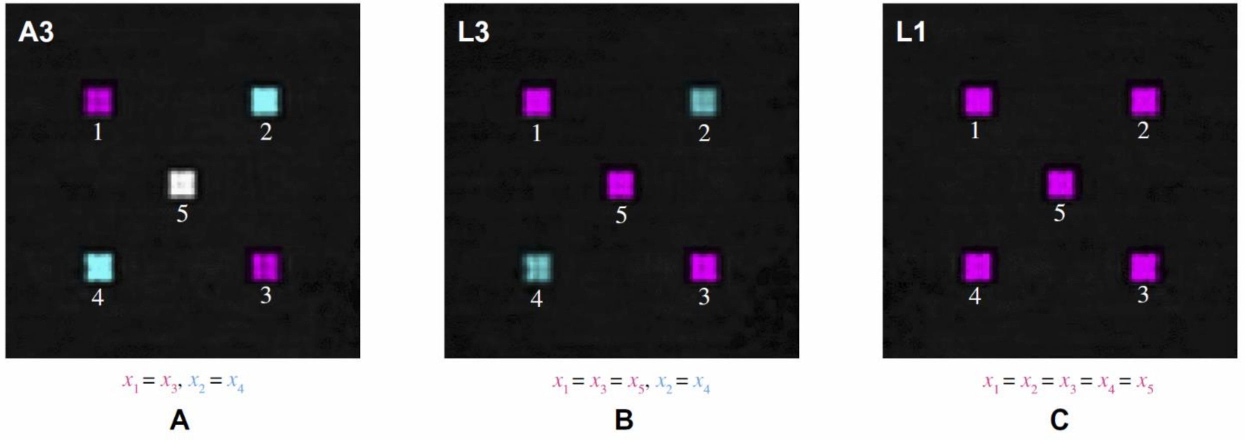
Pattern	Quotient network dynamics (right-hand side is mod2π)	Transverse perturbations
A1: $x_1 = x_2 = x_3 = x_4$	$x_1^{t+1} = [(\beta - \sigma)\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_5^{t+1} = [(\beta - 4\sigma)\mathcal{I}(x_5^t) + 4\sigma\mathcal{I}(x_1^t) + \delta]$	$\eta^{t+1} = [\beta + \sigma\lambda]\, D\mathcal{I}(x_1^t)\eta^t,\ \lambda = -3, -5$
A2: $x_1 = x_2 \quad \& \quad x_3 = x_4$	$x_1^{t+1} = [(\beta - 2\sigma)\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_4^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_2^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_2^t) + 3\sigma\mathcal{I}(x_1^t) + \delta]$ $x_4^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_4^t) + 3\sigma\mathcal{I}(x_1^t) + \delta]$	$\eta_1^{t+1} = [\beta - 4\sigma]\, D\mathcal{I}(x_1^t)\eta_1^t - \sigma D\mathcal{I}(x_3^t)\eta_3^t$ $\eta_3^{t+1} = -\sigma D\mathcal{I}(x_1^t)\eta_1^t + [\beta - 4\sigma]\, \sigma D\mathcal{I}(x_3^t)\eta_3^t$
A3: $x_1 = x_3 \quad \& \quad x_2 = x_4$	$x_1^{t+1} = [(\beta - 2\sigma)\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_2^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_2^{t+1} = [(\beta - 2\sigma)\mathcal{I}(x_2^t) + \sigma\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_5^{t+1} = [(\beta - 4\sigma)\mathcal{I}(x_5^t) + 2\sigma\mathcal{I}(x_1^t) + 2\sigma\mathcal{I}(x_2^t) + \delta]$	$\eta^{t+1} = [\beta + \lambda\sigma]\, D\mathcal{I}(x_1^t)\eta^t,\ \lambda = -3$
A4: $x_1 = x_3$	$x_1^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_2^t) + \sigma\mathcal{I}(x_4^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_2^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_2^t) + 2\sigma\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_4^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_4^t) + 2\sigma\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_5^t) + \delta]$ $x_5^{t+1} = [(\beta - 4\sigma)\mathcal{I}(x_5^t) + 2\sigma\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_2^t) + \sigma\mathcal{I}(x_4^t) + \delta]$	$\eta^{t+1} = [\beta + \lambda\sigma]\, D\mathcal{I}(x_1^t)\eta^t,\ \lambda = -3$
L1: $x_1 = x_2 = \dots = x_5$	$x_1^{t+1} = [\beta\mathcal{I}(x_1^t) + \delta]$	$\eta^{t+1} = [\beta + \sigma\lambda]\, D\mathcal{I}(x_1^t)\eta^t,\ \lambda = -3, -5$
L3: $x_1 = x_3 = x_5 \,\&\, x_2 = x_4$	$x_1^{t+1} = [(\beta - 2\sigma)\mathcal{I}(x_1^t) + 2\sigma\mathcal{I}(x_2^t) + \delta]$ $x_2^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_2^t) + 3\sigma\mathcal{I}(x_1^t) + \delta]$	$\eta_1^{t+1} = [\beta + \sigma\lambda]\, D\mathcal{I}(x_1^t)\eta_1^t,\ \lambda = -3, -5$ $\eta_2^{t+1} = [\beta + \sigma\lambda]\, D\mathcal{I}(x_2^t)\eta_2^t,\ \lambda = -3$
L4: $x_1 = x_3 = x_5$	$x_1^{t+1} = [(\beta - 2\sigma)\mathcal{I}(x_1^t) + \sigma\mathcal{I}(x_2^t) + \sigma\mathcal{I}(x_4^t) + \delta]$ $x_2^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_2^t) + 3\sigma\mathcal{I}(x_1^t) + \delta]$ $x_4^{t+1} = [(\beta - 3\sigma)\mathcal{I}(x_4^t) + 3\sigma\mathcal{I}(x_1^t) + \delta]$	$\eta^{t+1} = [\beta + \sigma\lambda]\, D\mathcal{I}(x_1^t)\eta^t,\ \lambda = -3, -5$



Experimental phase space plots with lines connecting successive iterates. (A) Three clusters (A3). (B) Two clusters (L3). (C) One cluster (L1).

Experimental Validation





Experimental patterns of light intensity of different clusters in the five-node network. (A to C)

Conclusions and open questions

- Symmetries are commonly found in many real networks
- The synchronization dynamics of these networks is influenced by their symmetries. In particular: the formation of synchronization clusters and their stability
- Given a network topology, several synchronization patterns are in general possible and in general they can be studied through a sequence of symmetry breaking bifurcation from a single maximally synchronous pattern